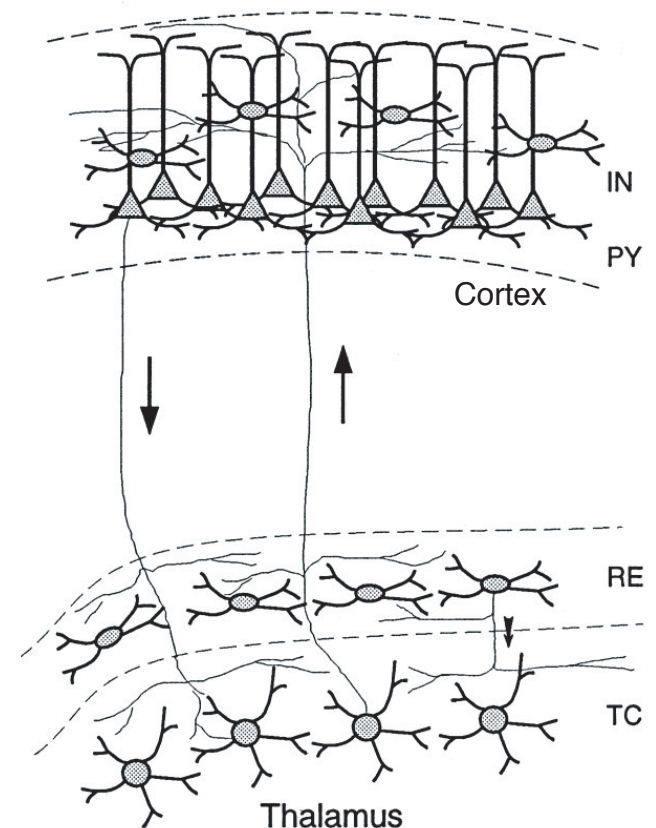


Mathematical Neuroscience: from neurons to networks



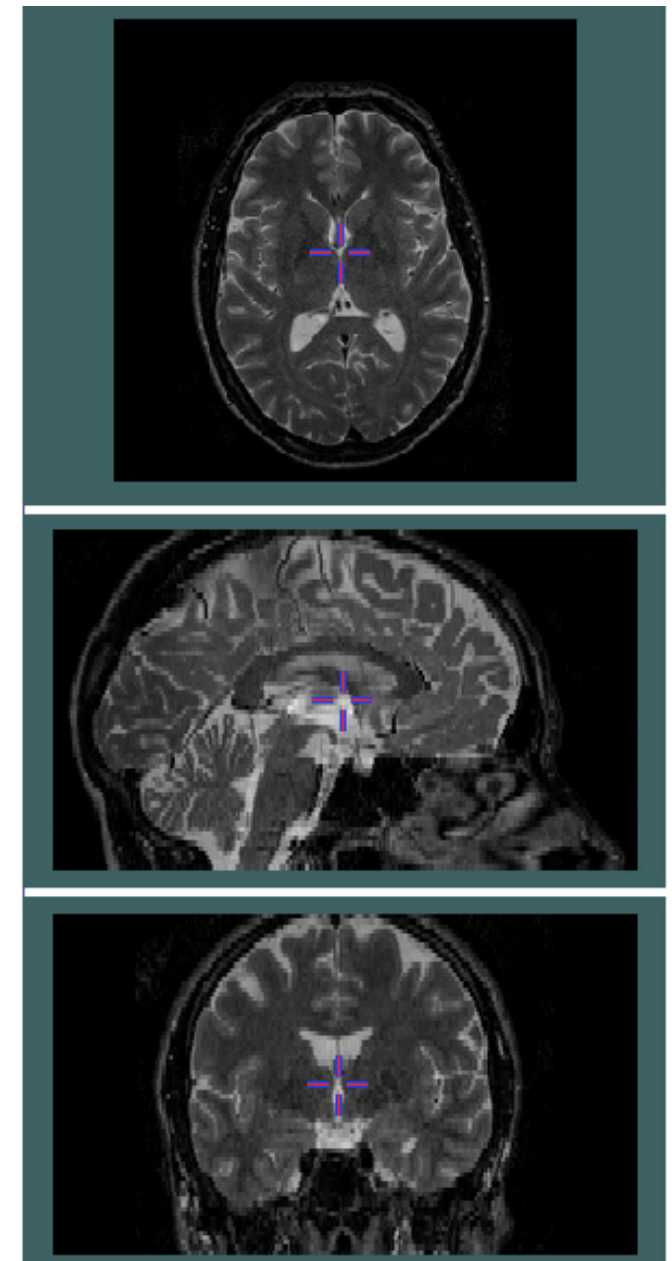
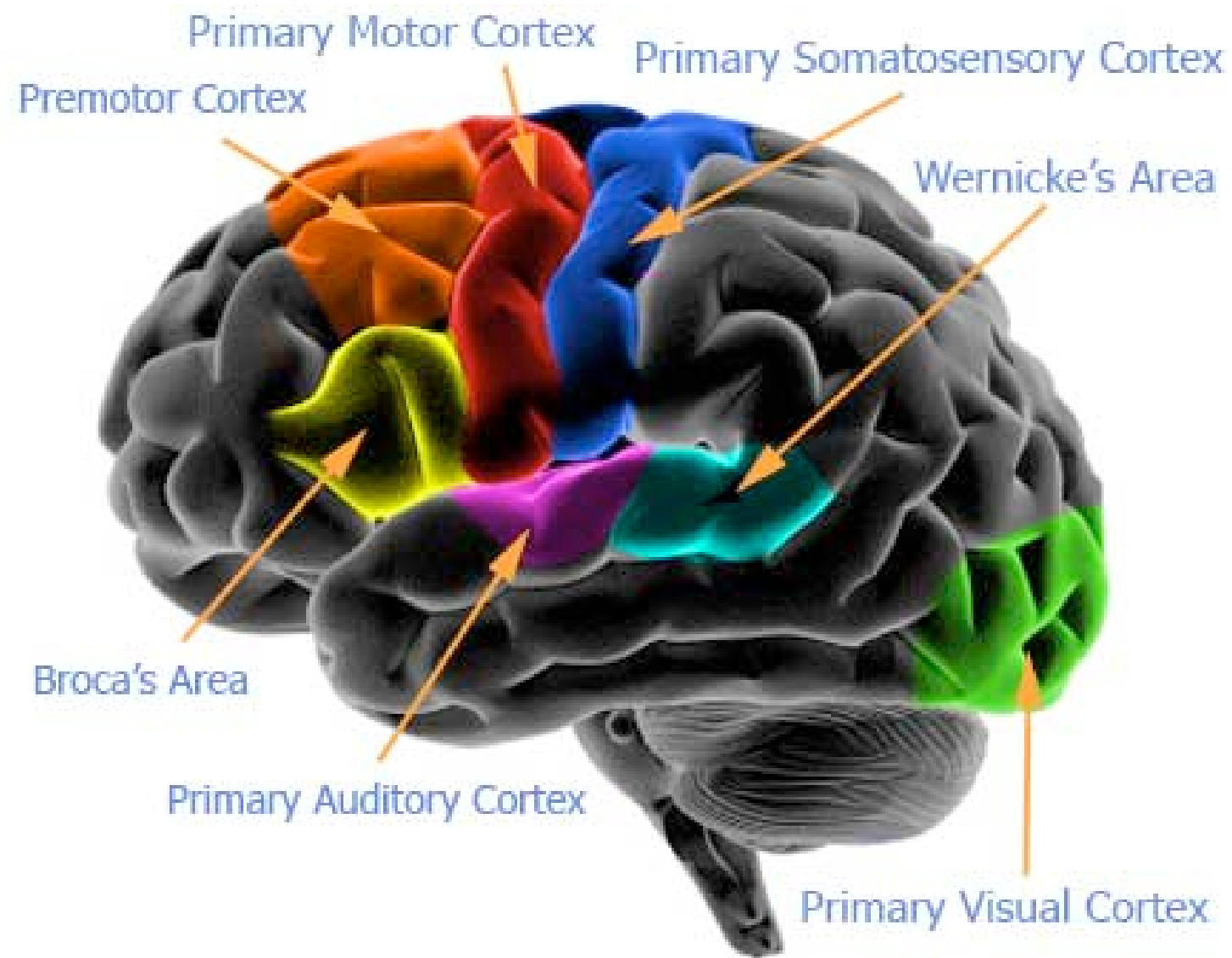
Steve
Coombes



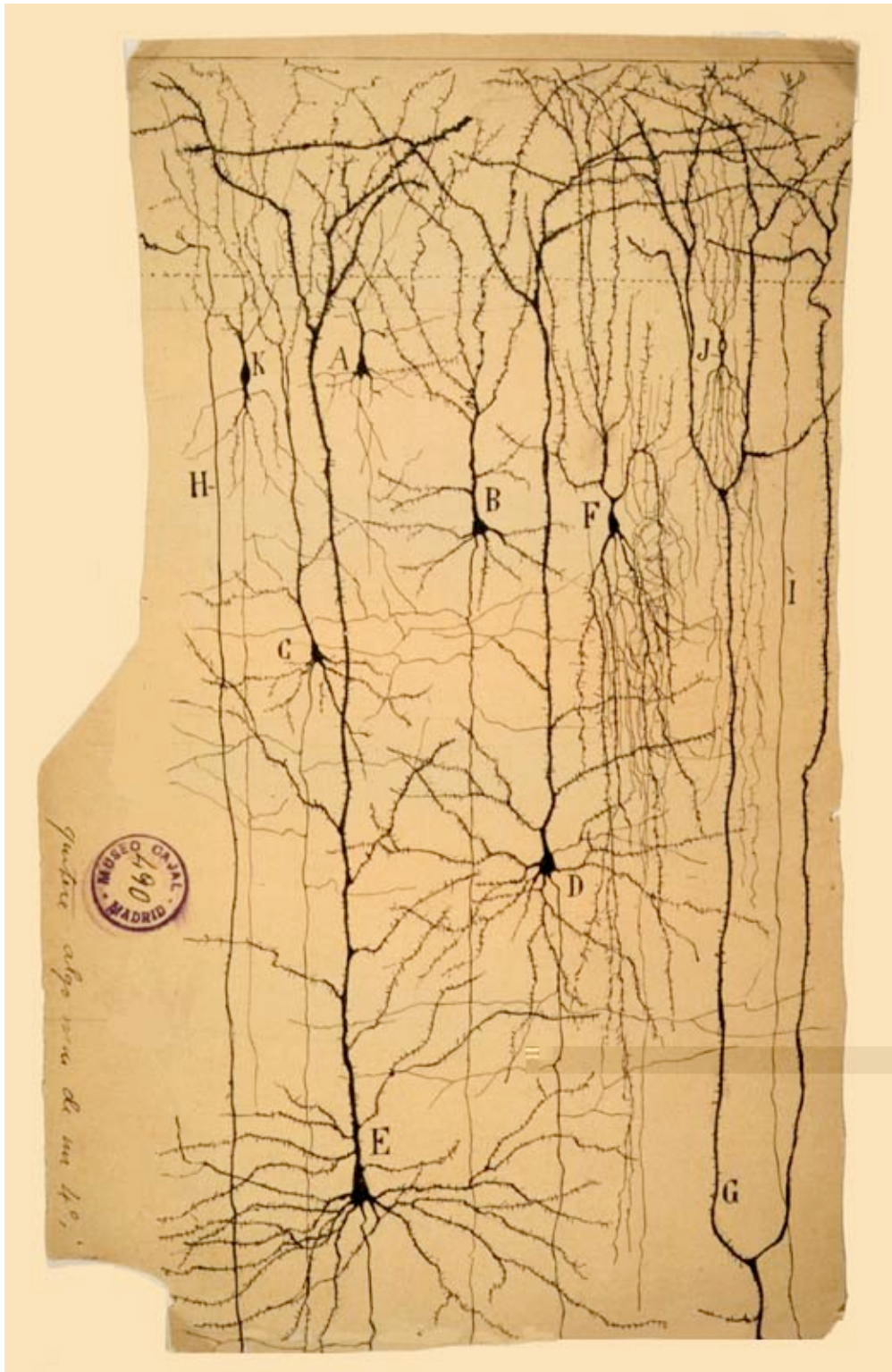
The University of
Nottingham

School of Mathematical
Sciences

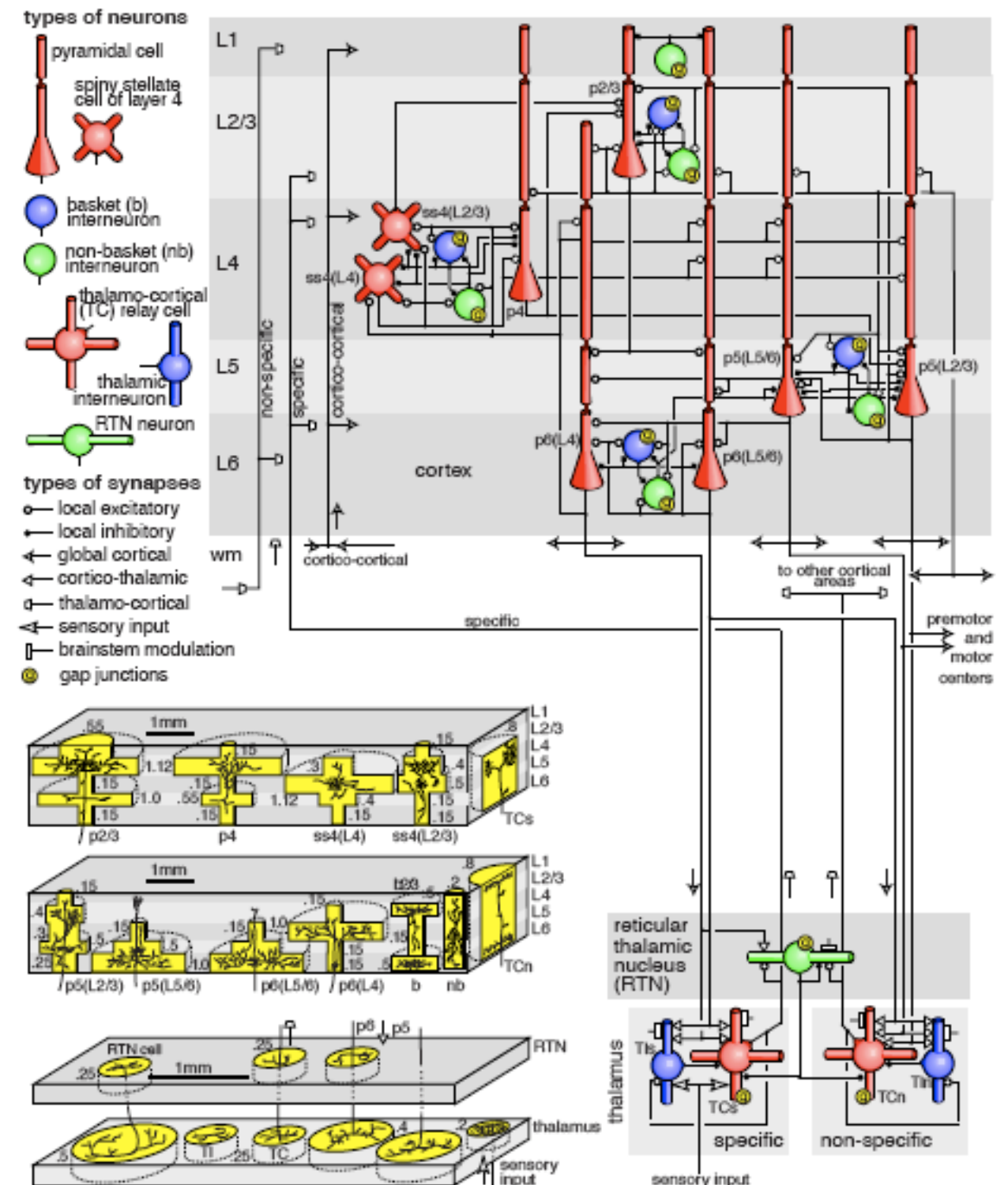
Brain and Cortex



Principal cells and interneurons

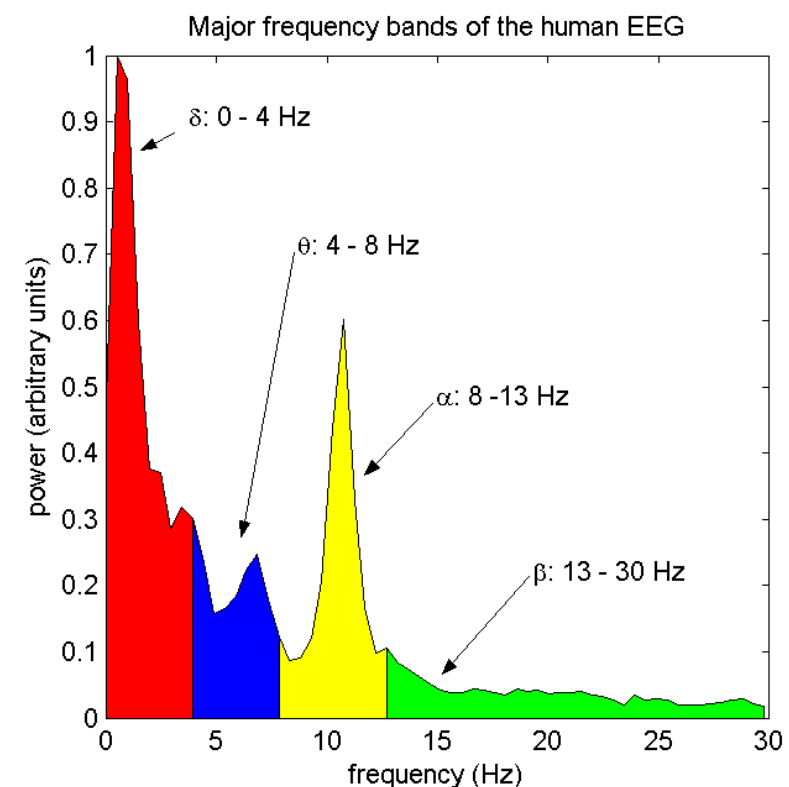
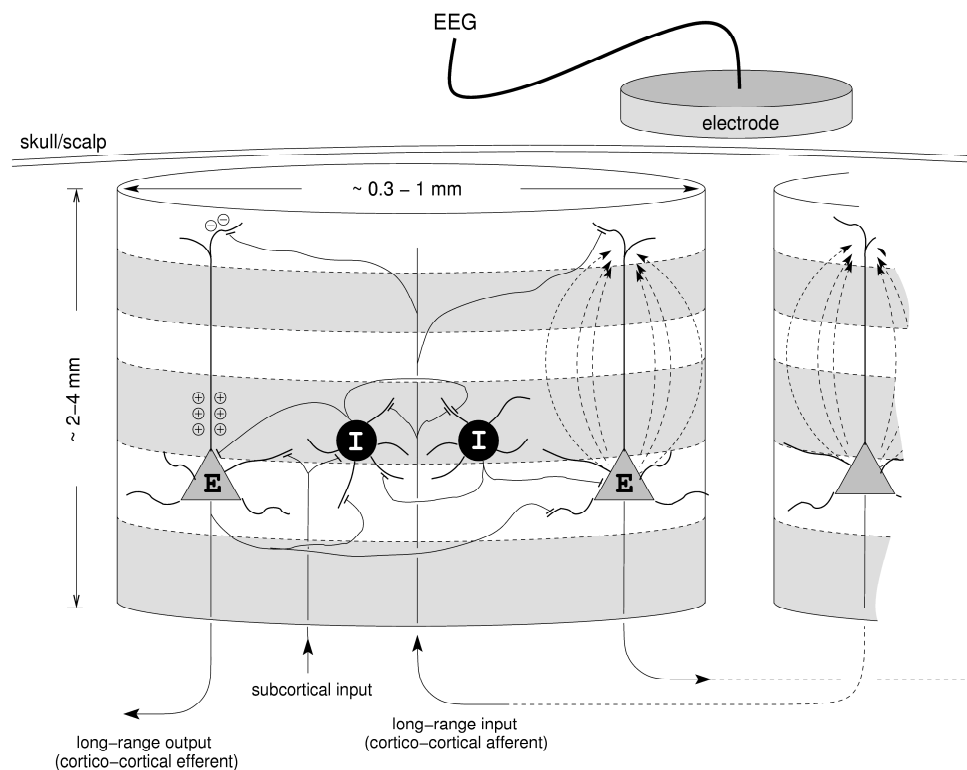
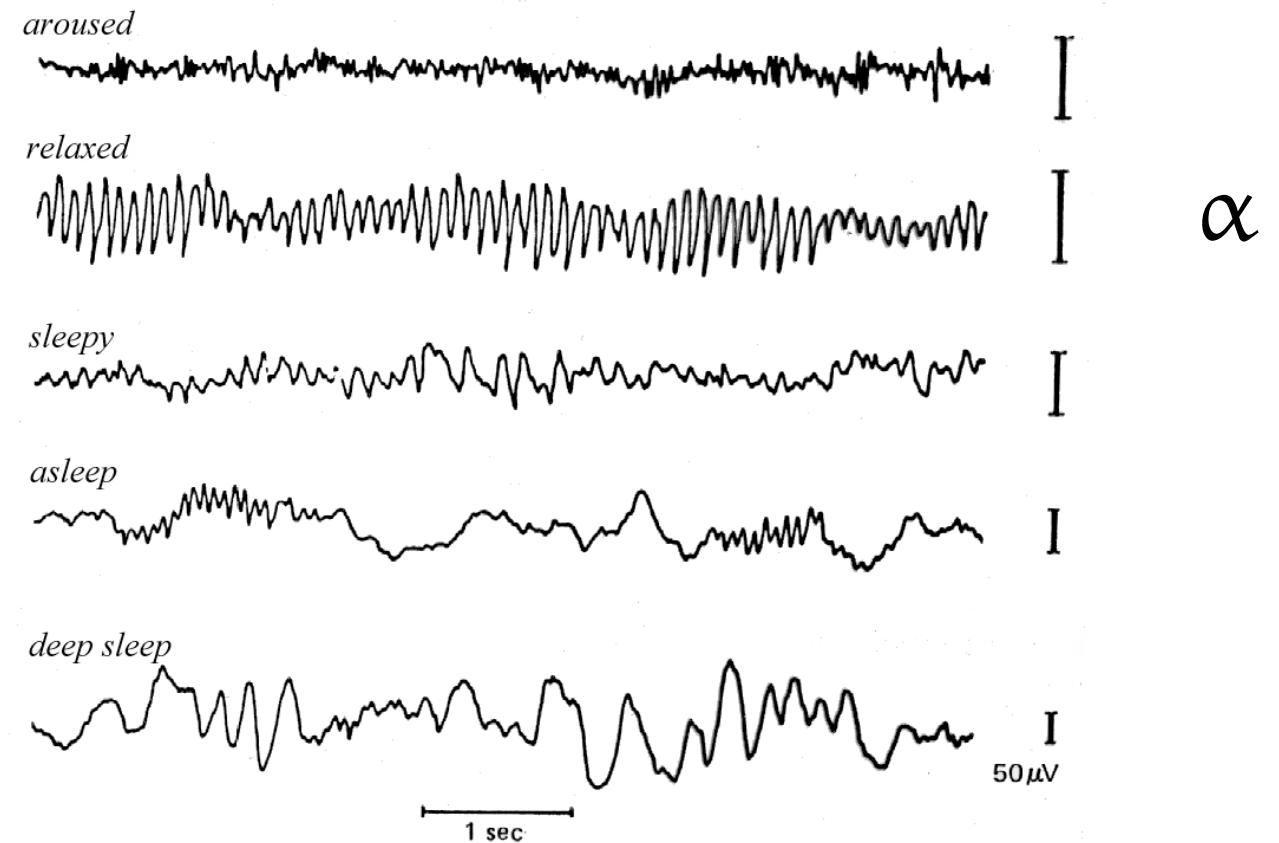


Santiago Ramón y Cajal
1900



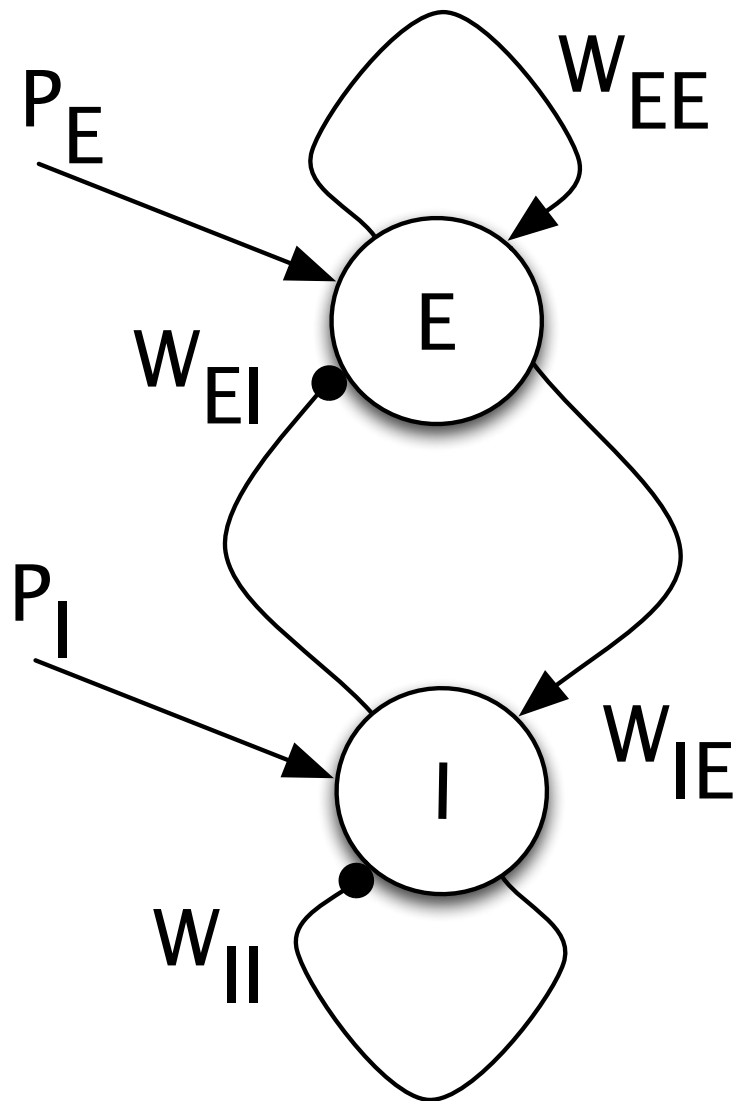
Eugene Izhikevich
2008

Electroencephalogram (EEG) power spectrum

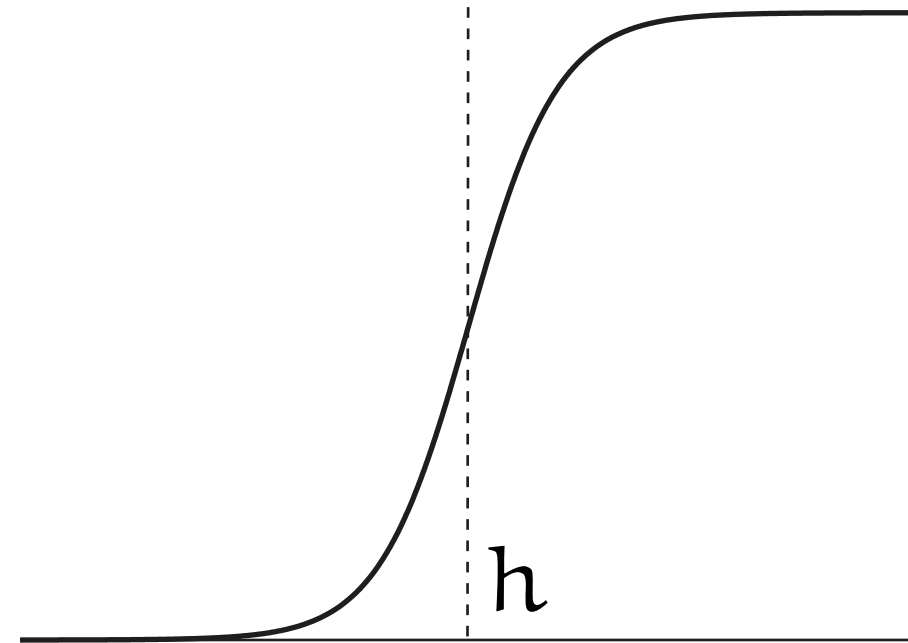


EEG records the activity of $\sim 10^6$ pyramidal neurons.

Population model



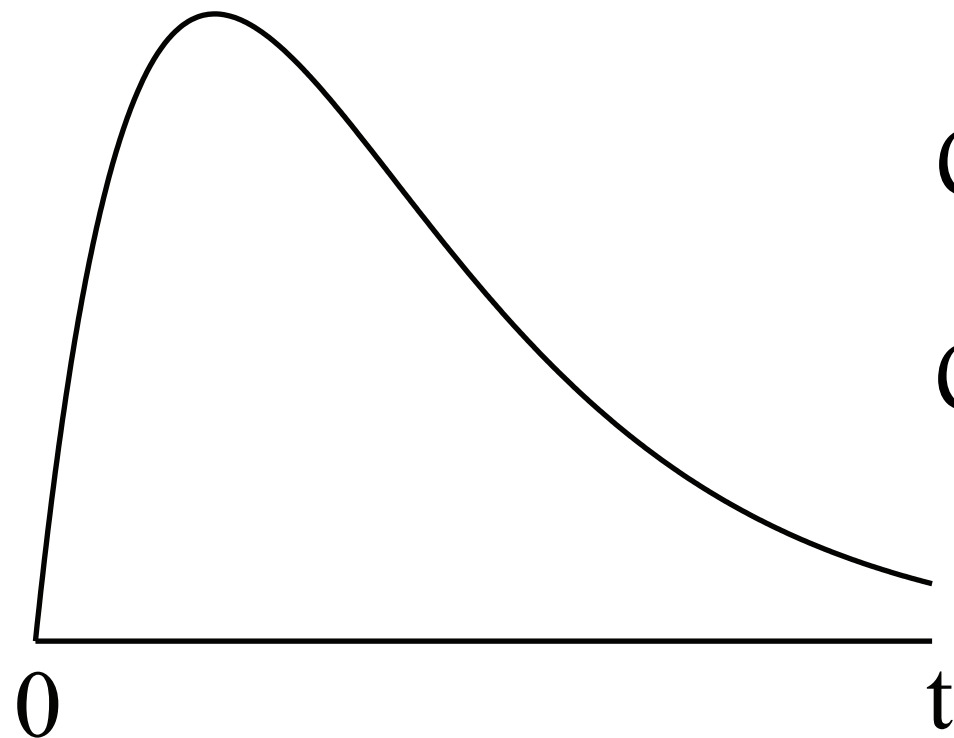
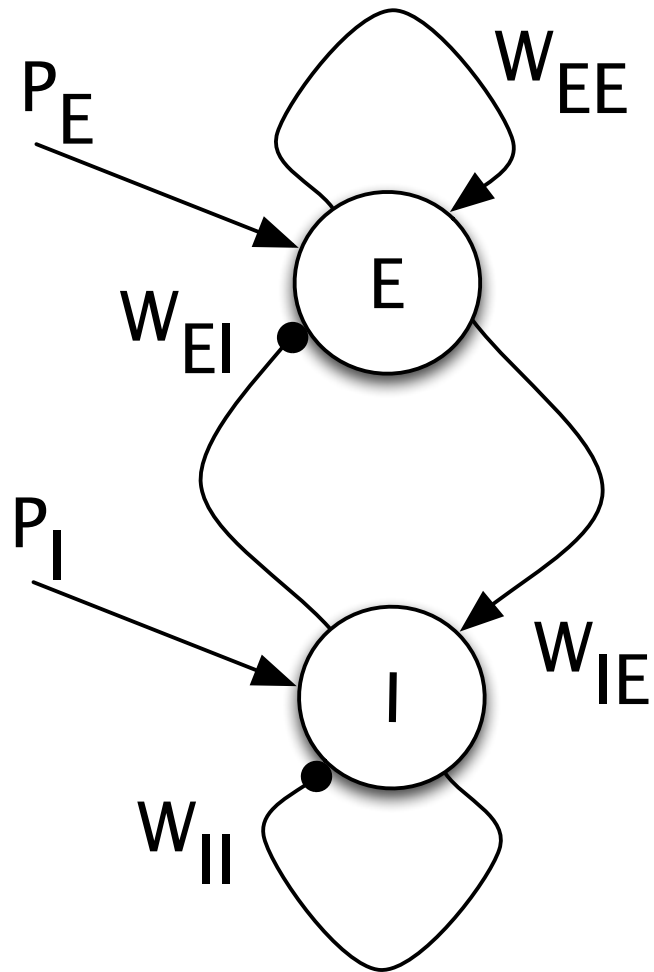
Firing rate activity $f(E)$



Firing rate activity $f(I)$

$$\dot{E} = -\frac{E}{\tau_E} + W_{EE}g_{EE}(A^+ - E) + W_{EI}g_{EI}(A^- - E) + P_E$$

$$\dot{I} = -\frac{I}{\tau_I} + W_{II}g_{II}(A^- - I) + W_{IE}g_{IE}(A^+ - I) + P_I$$



$$\eta(t) = \alpha^2 t e^{-\alpha t}$$

$$Q\eta = \delta$$

$$Q = \left(1 + \frac{1}{\alpha} \frac{d}{dt}\right)^2$$

$$Qg_{jE} = f(E)$$

$$Qg_{jI} = f(I)$$

**Steady state
approximation**

$$E = E(g_{EE}, g_{EI})$$

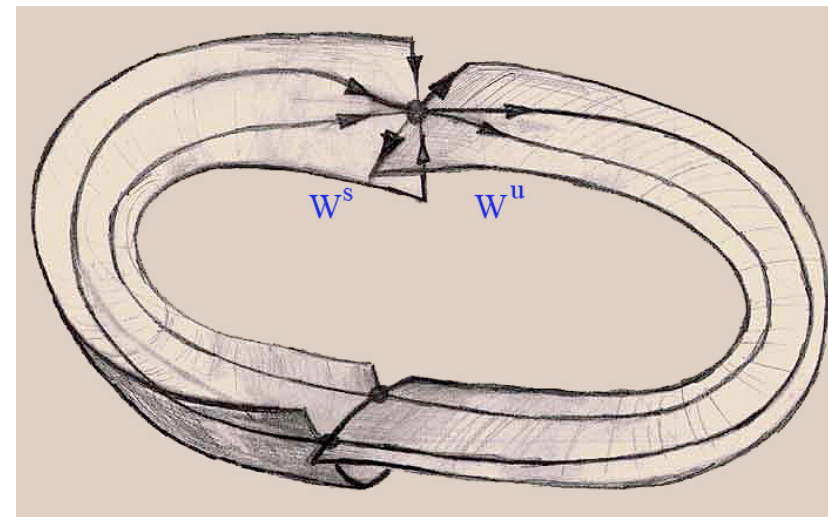
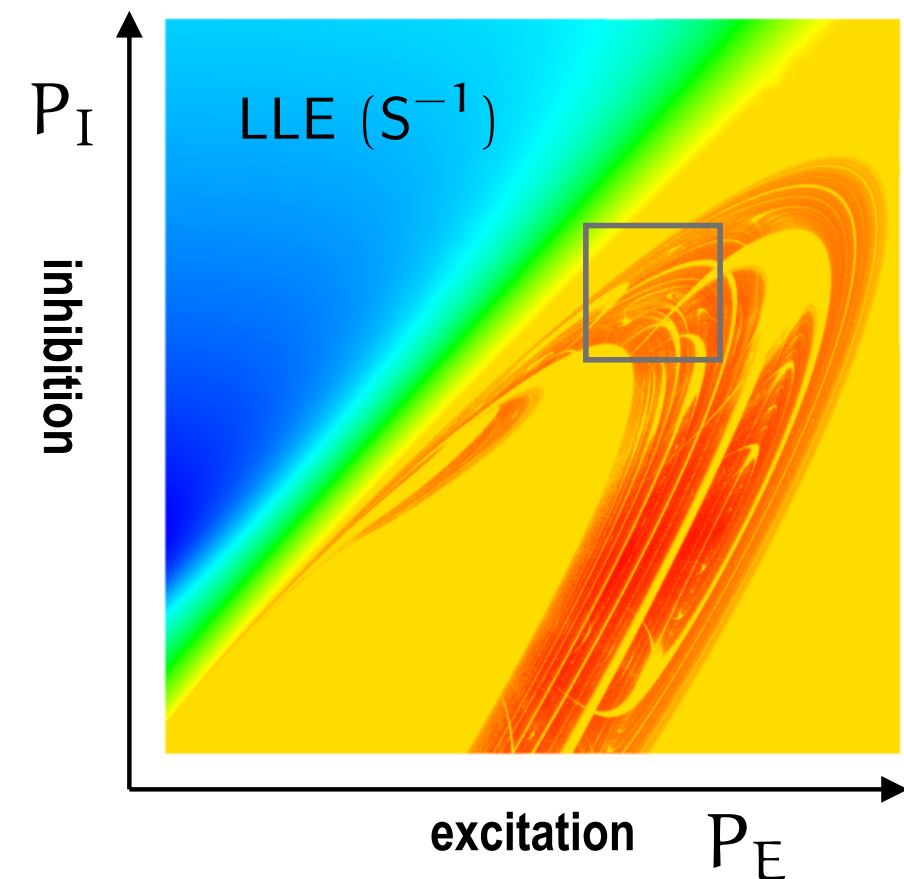
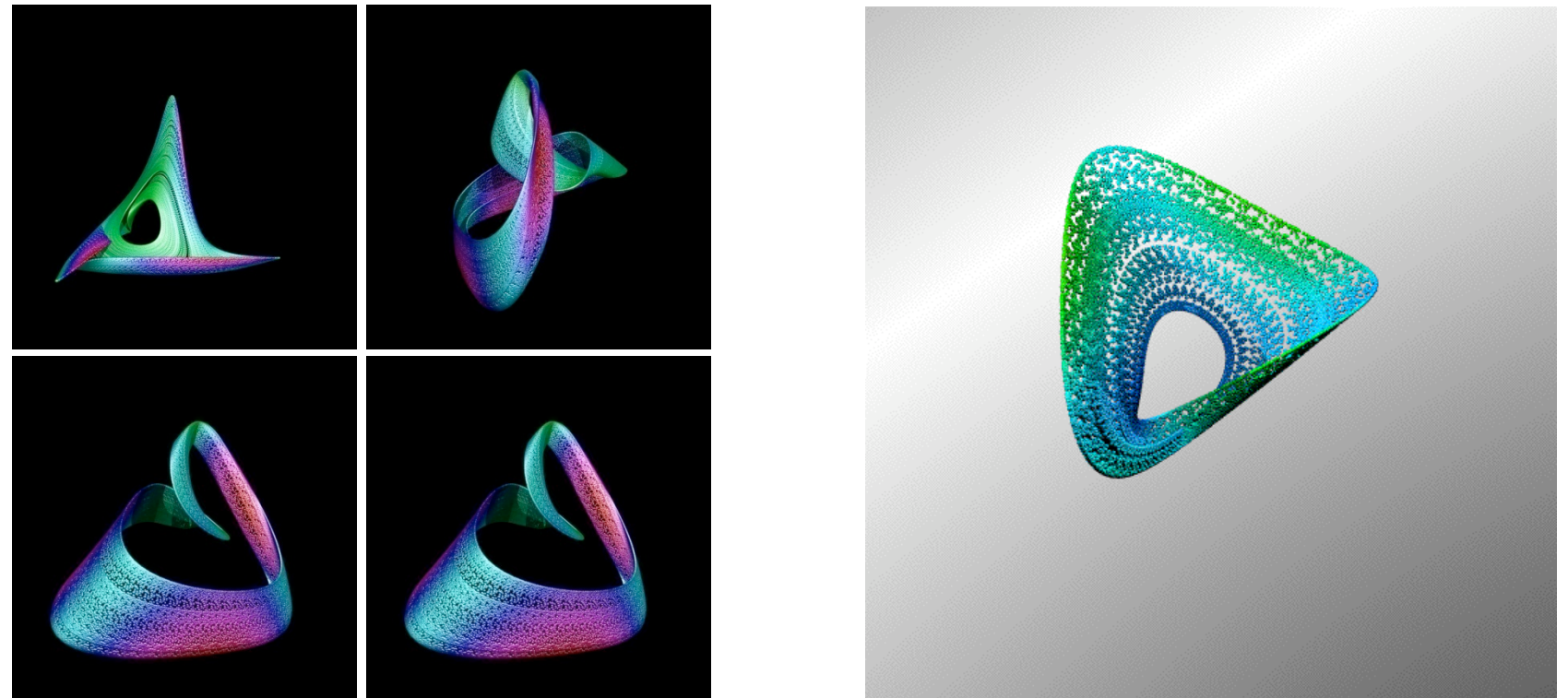
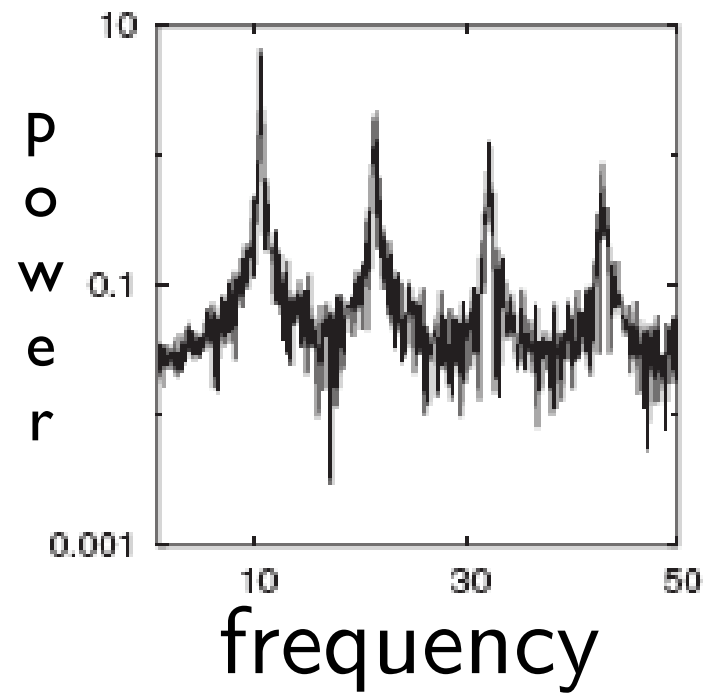
$$I = I(g_{II}, g_{IE})$$

$$Qg = f$$

$$f = f(\{g\})$$

$$g = \eta * f$$

Alphoid chaos (10 D)

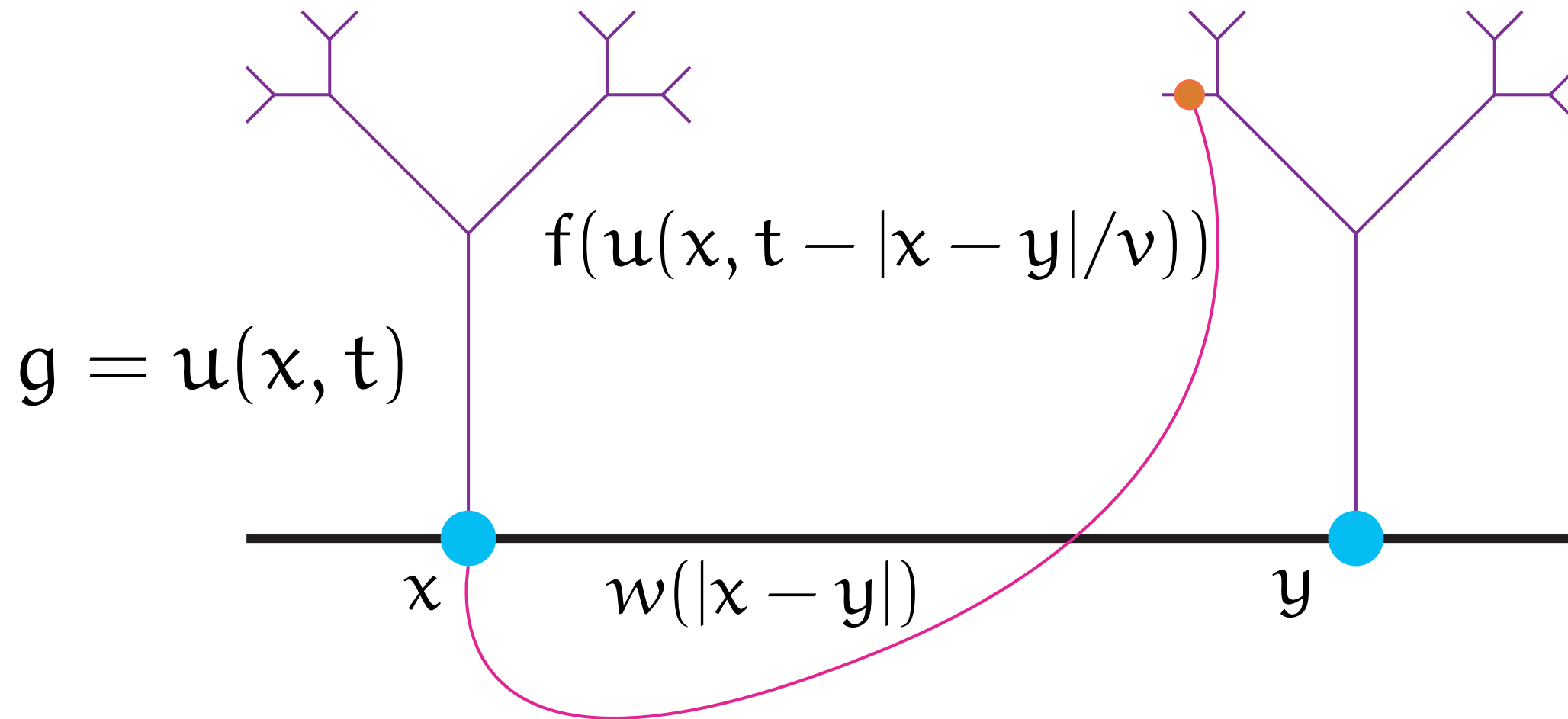


Shilnikov saddle-node route to chaos
van Veen and Liley, PRL, **97**, 208101 (2006)

Spatially extended models

$$g = w \otimes \eta * f$$

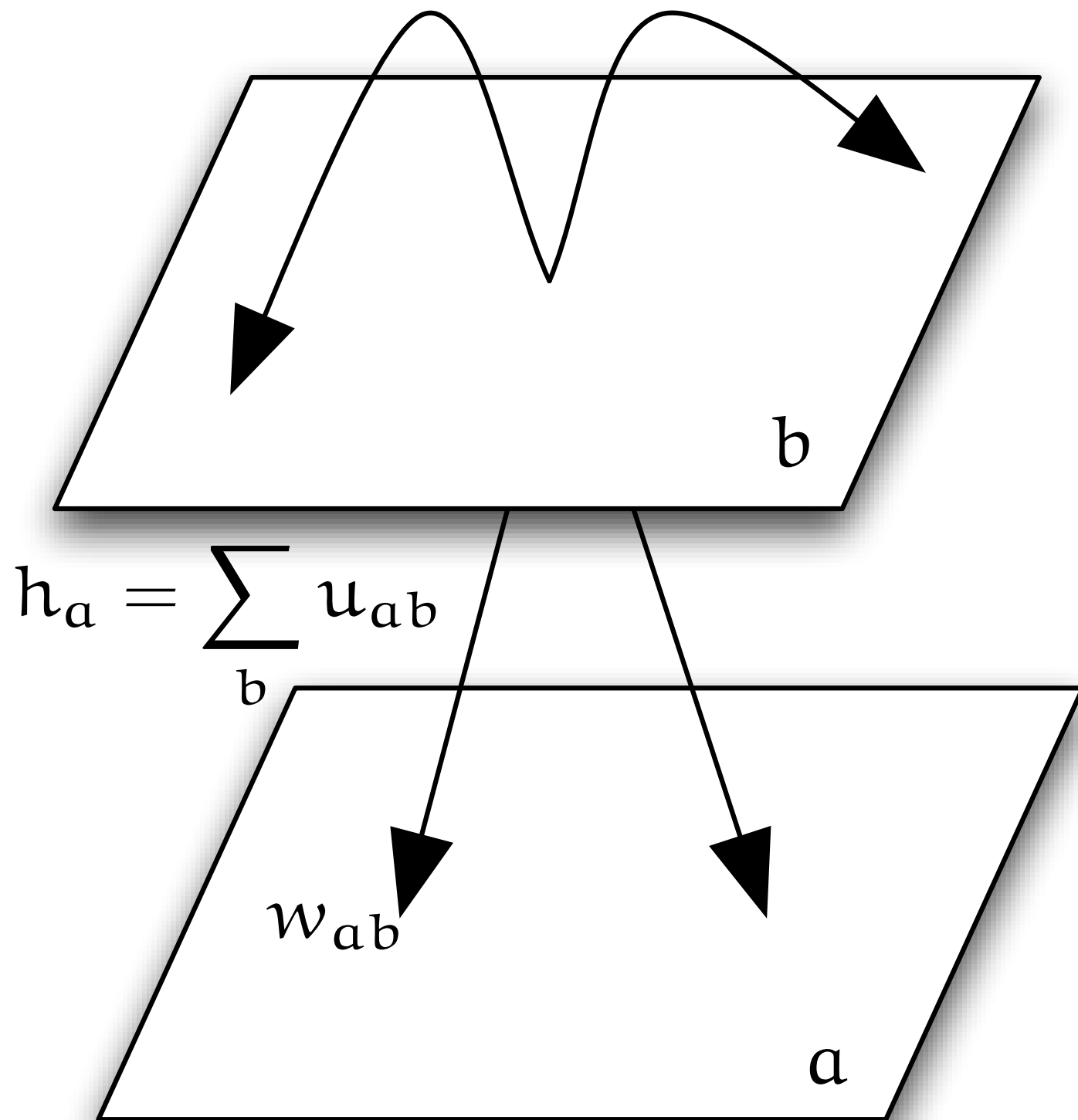
Simplest neural field model: Wilson-Cowan ('72), Amari ('77)



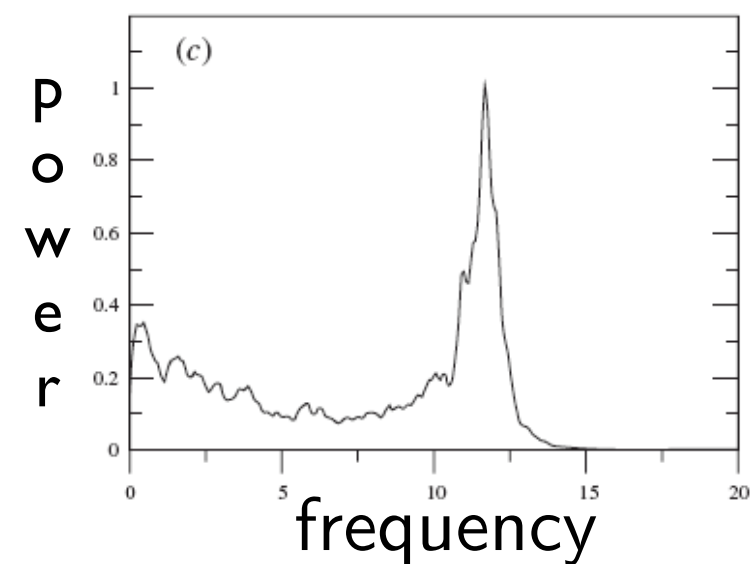
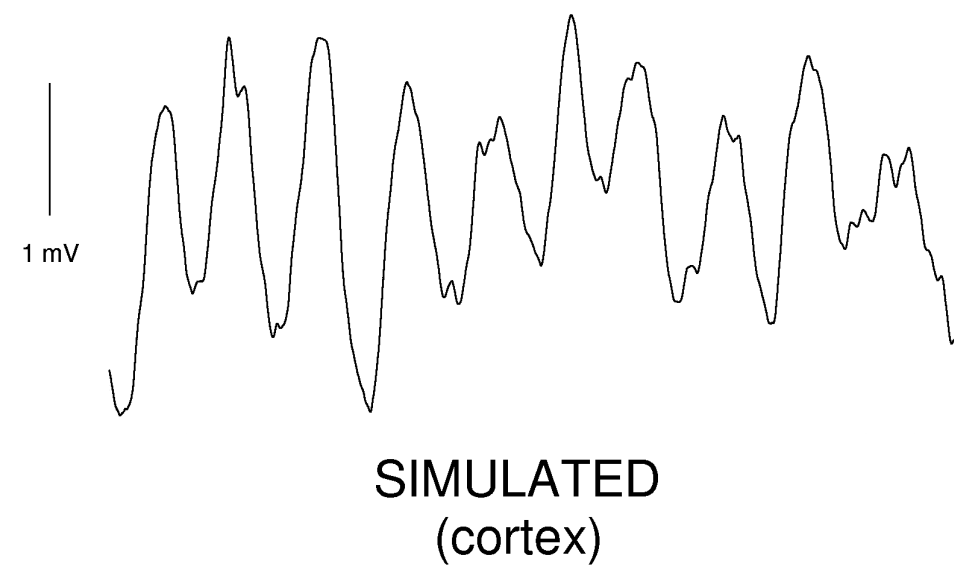
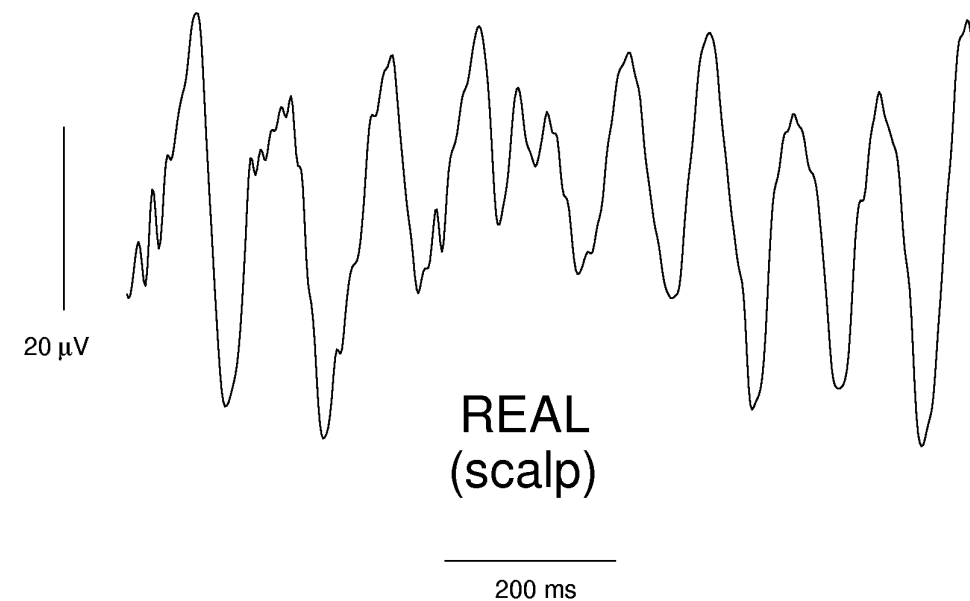
$$u(x, t) = \int_{-\infty}^{\infty} dy \, w(y) \int_0^{\infty} ds \, \eta(s) \, f(u(x - y, t - s - |y|/v))$$

2D layers

$$u_{ab} = \eta_{ab} * \psi_{ab}$$

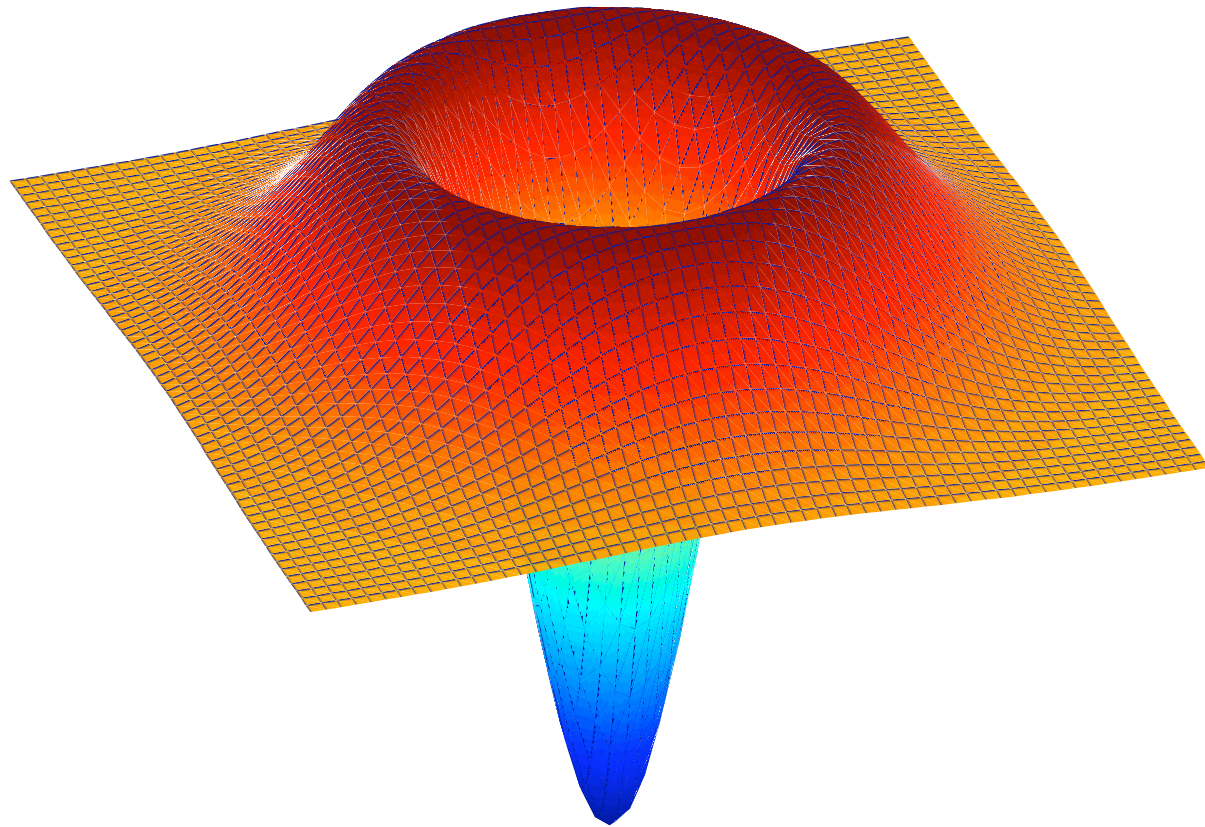


$$\psi_{ab}(\mathbf{r}, t) = \int_{\mathbb{R}^2} d\mathbf{r}' w_{ab}(\mathbf{r}, \mathbf{r}') f_b \circ h_b(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/v_{ab})$$



Turing instability analysis

E layer and I layer



$$e^{i\mathbf{k}\cdot\mathbf{r}} e^{\lambda t}$$

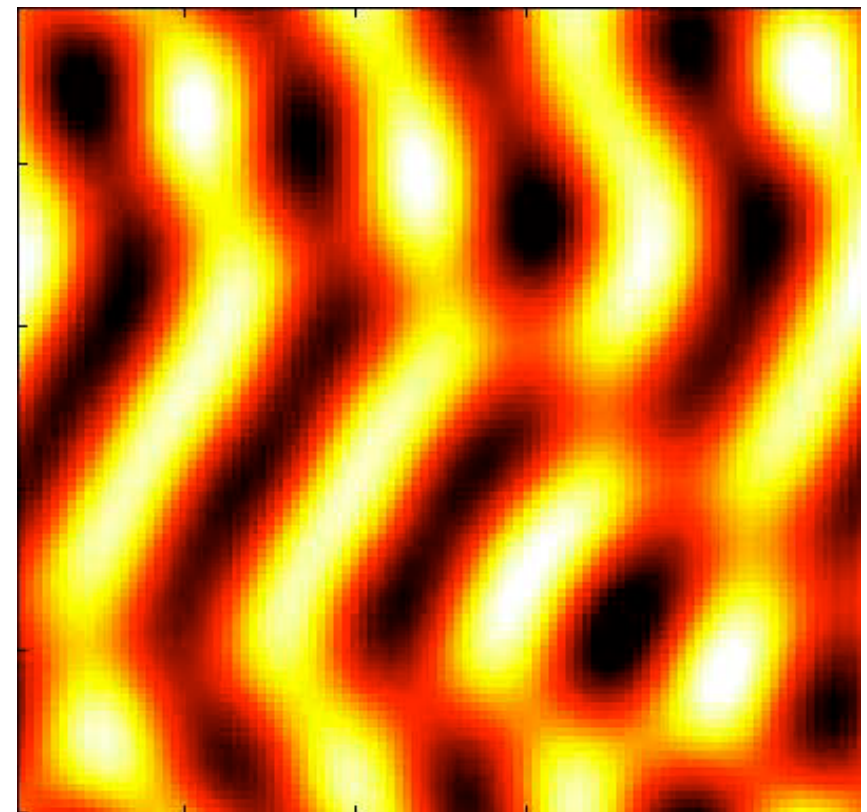
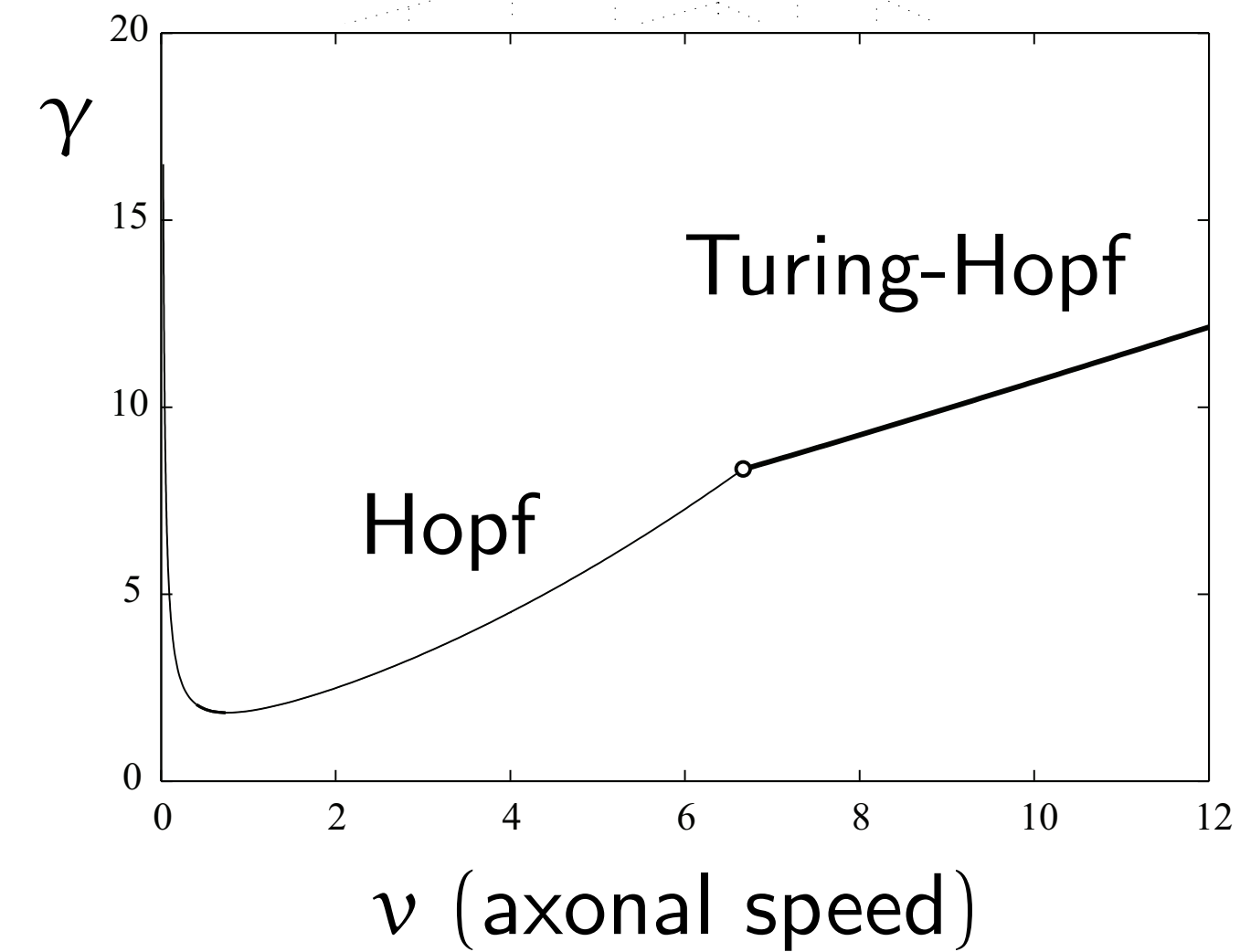
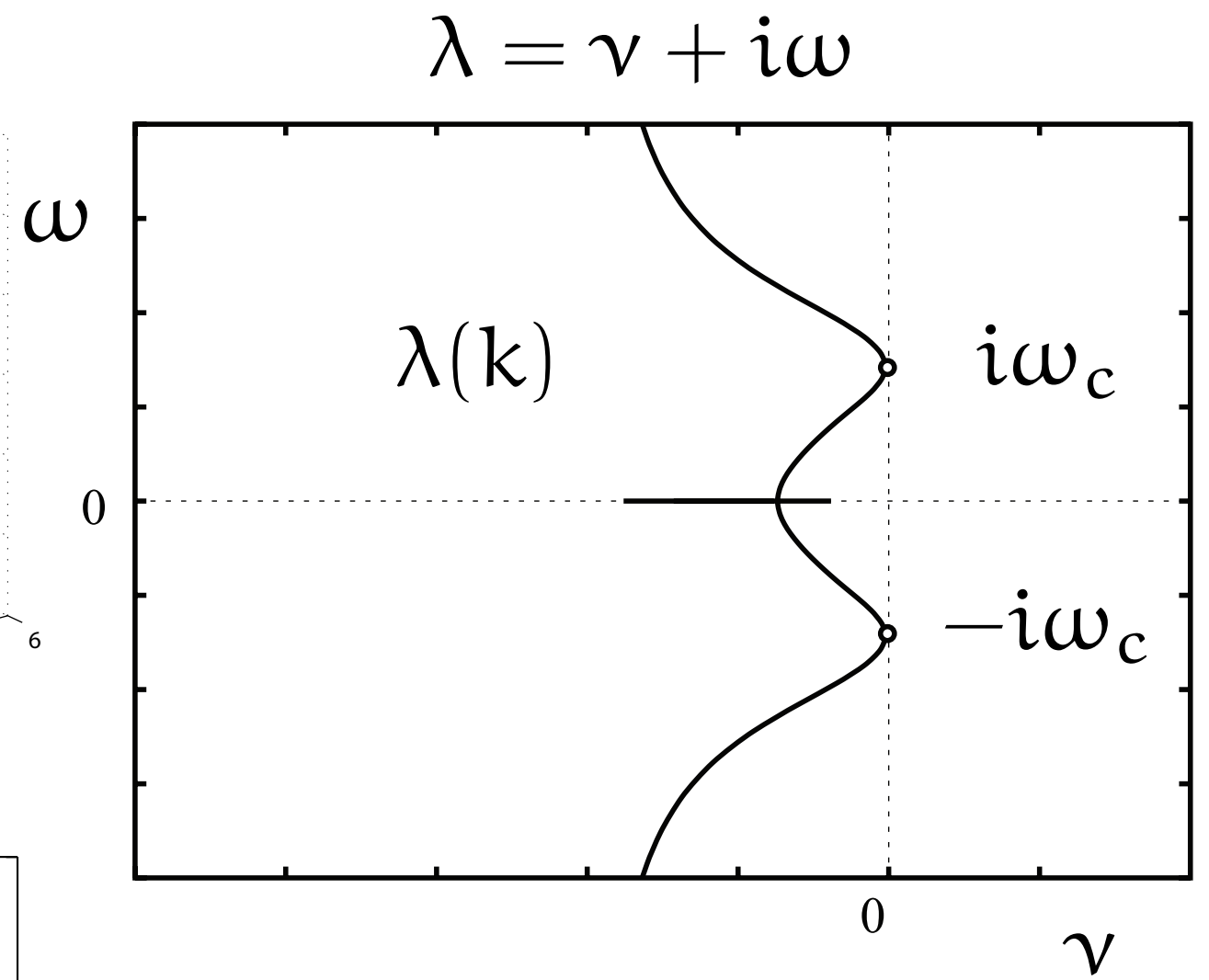
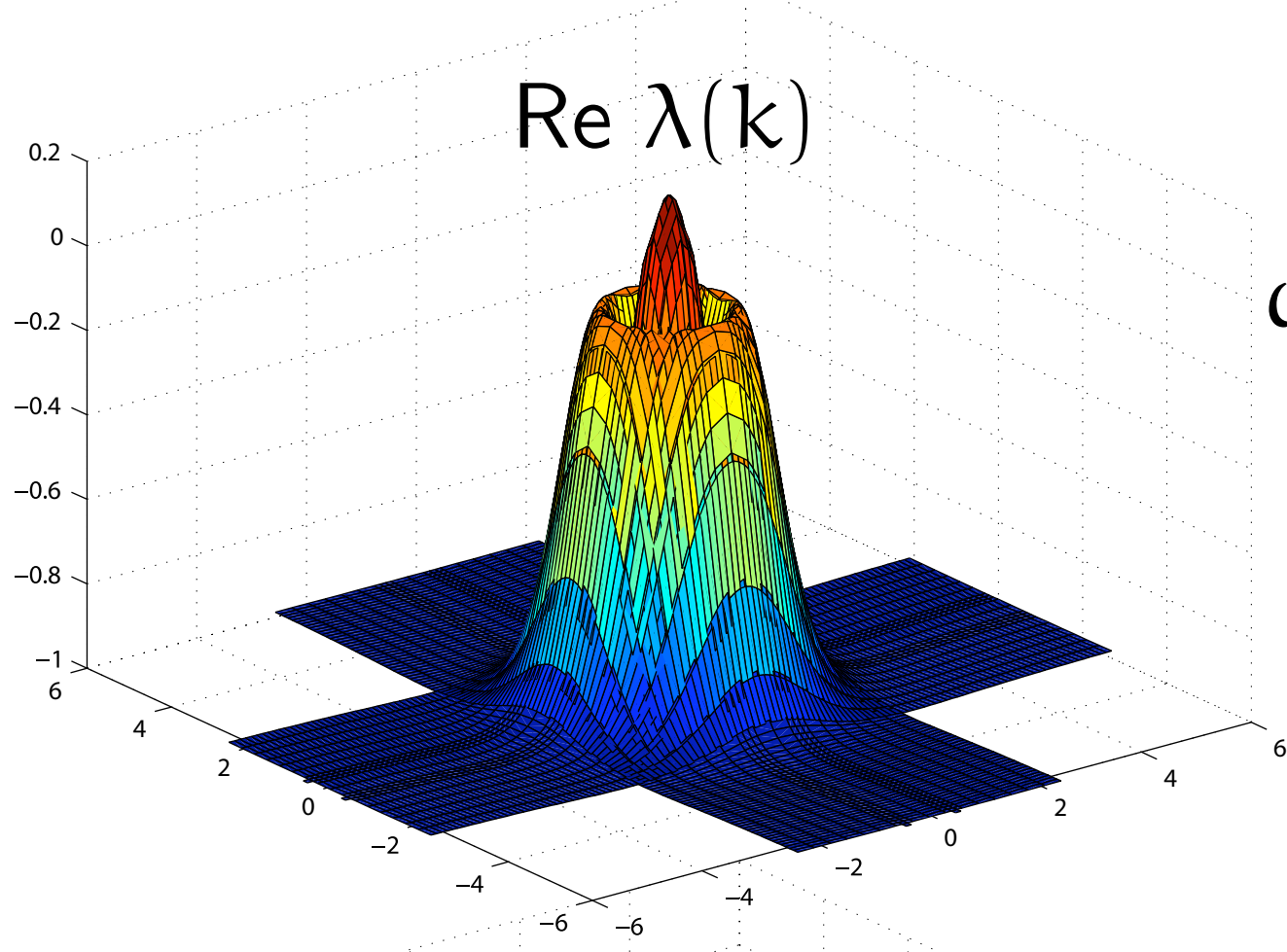
Continuous spectrum

$$\det (\mathcal{D}(\mathbf{k}, \lambda) - \mathbf{I}) = 0$$

$$[\mathcal{D}(\mathbf{k}, \lambda)]_{ab} = \tilde{\eta}_{ab}(\lambda) G_{ab}(\mathbf{k}, -i\lambda) \gamma_b$$

$$\tilde{\eta} = \text{LT } \eta \qquad G = \text{FLT } w(r) \delta(t - r/v) \qquad \gamma = f'(ss)$$

S Coombes et al., PRE, **76**, 05190 (2007)



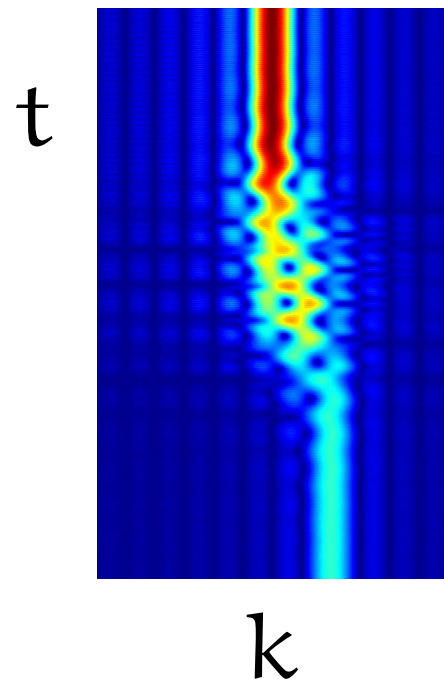
Amplitude Equations (one D)

Coupled mean-field Ginzburg–Landau equations describing a Turing–Hopf bifurcation with modulation group velocity of $O(1)$.

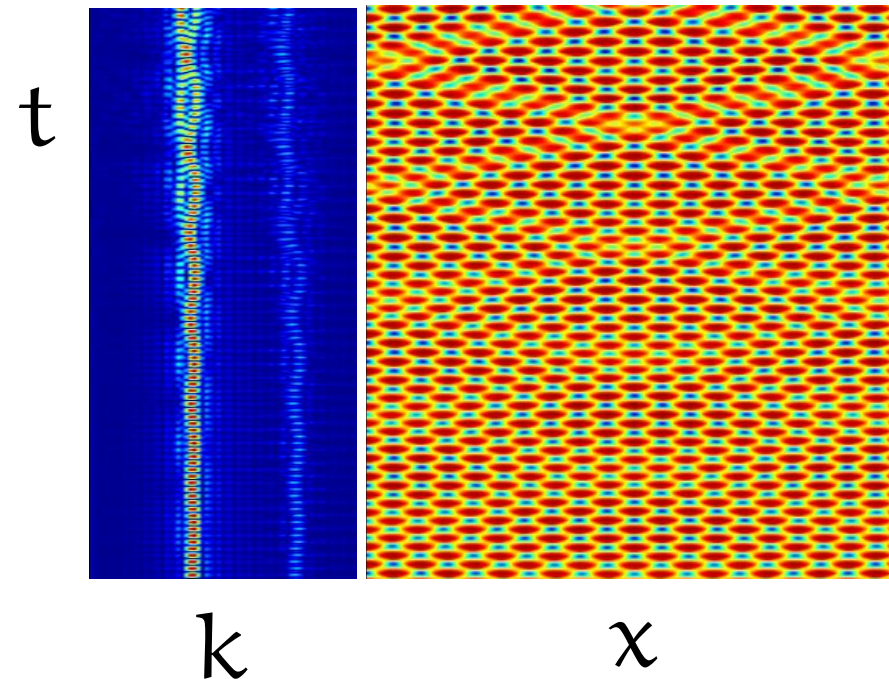
$$\frac{\partial A_1}{\partial \tau} = A_1 (a + b|A_1|^2 + c\langle |A_2|^2 \rangle) + d \frac{\partial^2 A_1}{\partial \xi_+^2}$$

$$\frac{\partial A_2}{\partial \tau} = A_2 (a + b|A_2|^2 + c\langle |A_1|^2 \rangle) + d \frac{\partial^2 A_2}{\partial \xi_-^2}$$

Benjamin–Feir (BF)

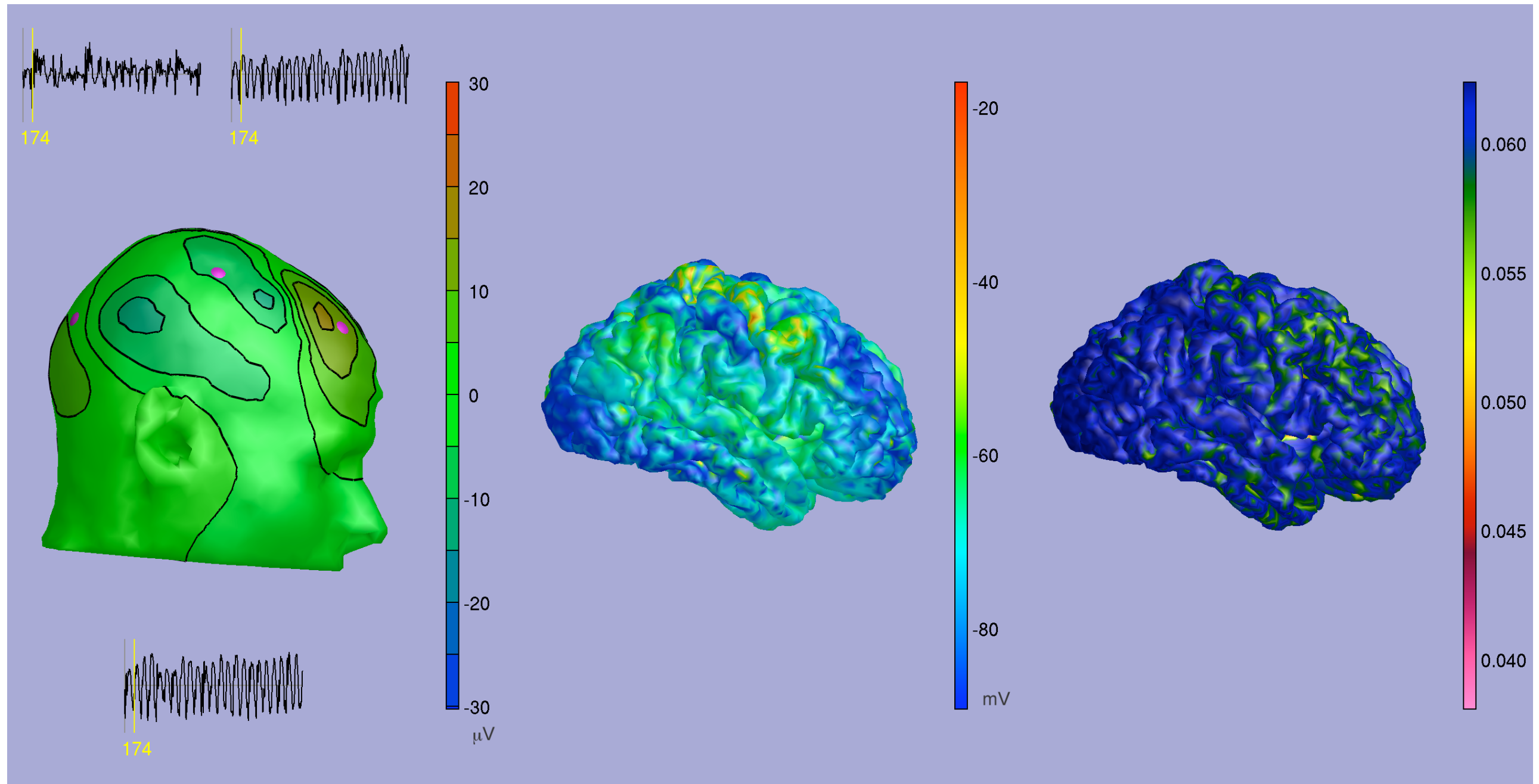


BF-Eckhaus instability



Coefficients in terms of integral transforms of w and η .

Applications to co-registered EEG/fMRI

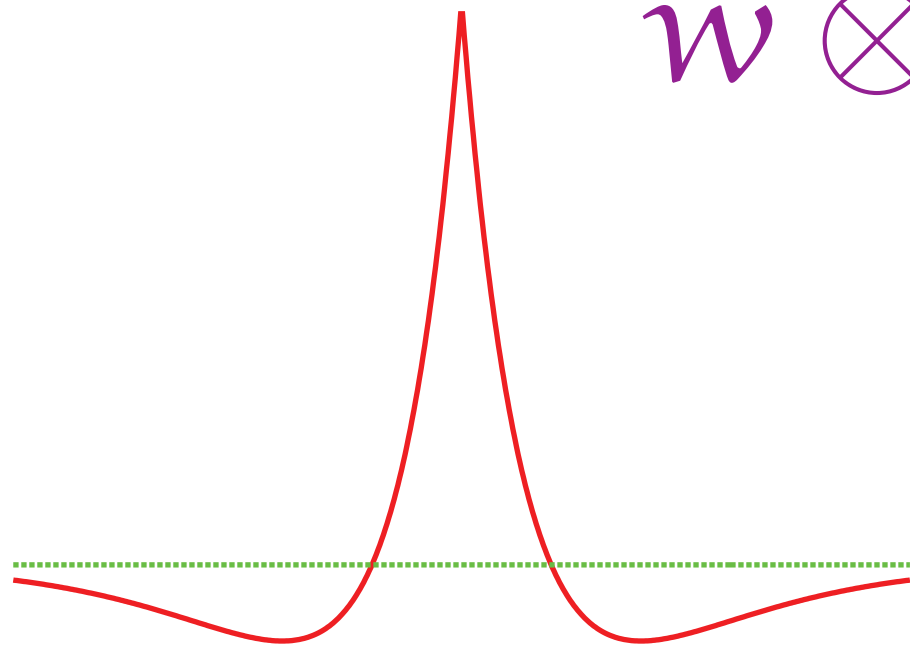


Bojak, I., Oostendorp, T. F., Reid, A. T., Kotter, R., 2009. Realistic mean field forward predictions for the integration of co-registered EEG/fMRI. BMC Neuroscience 10, L2.

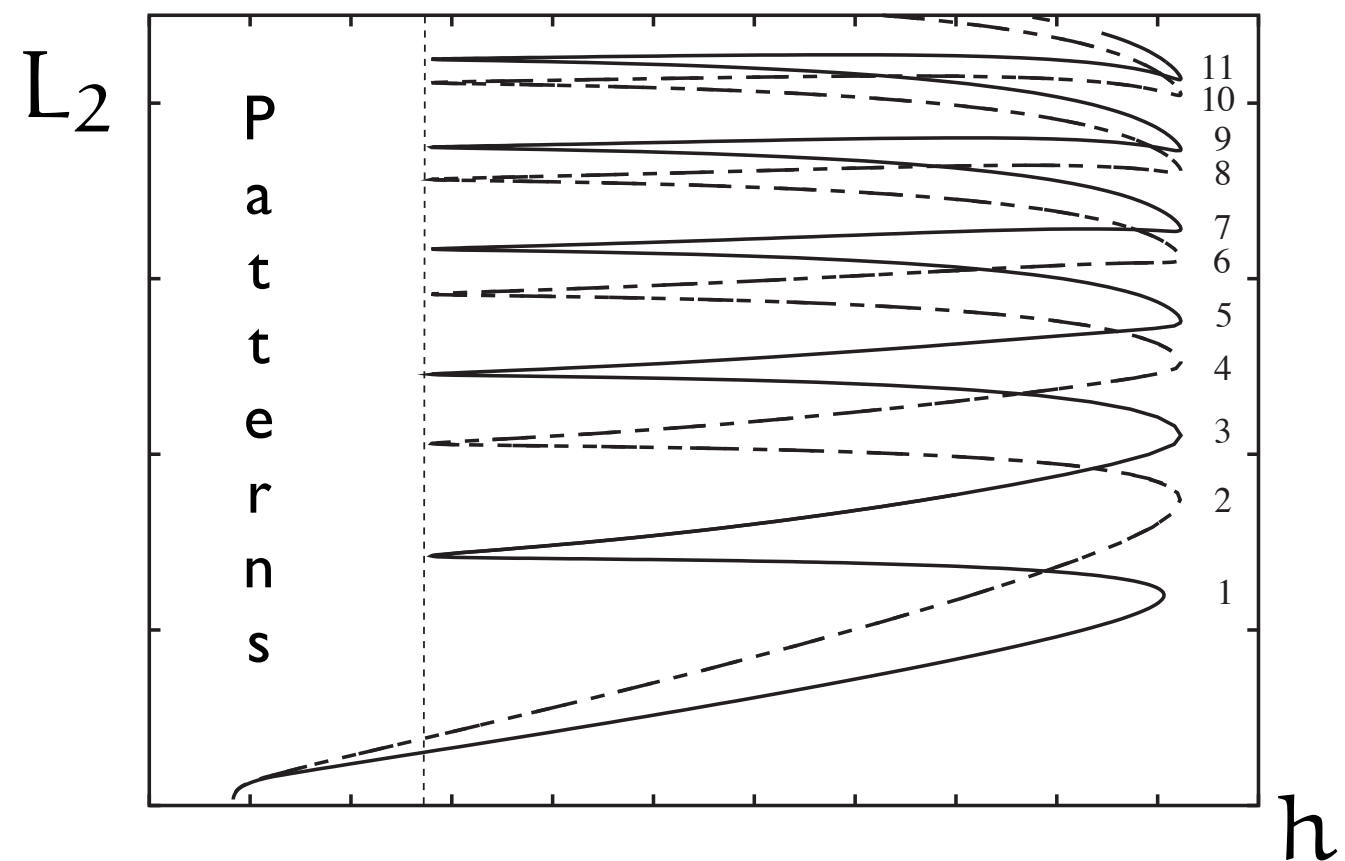
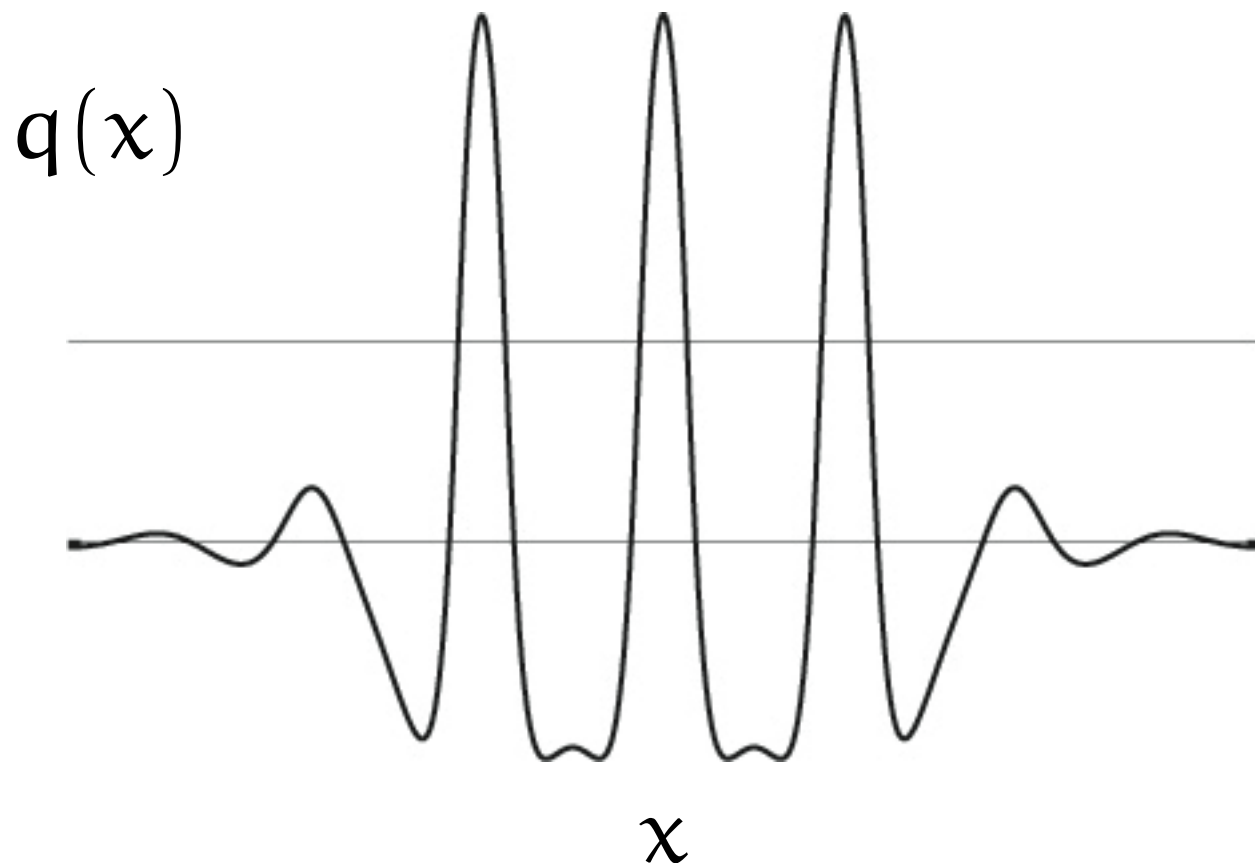
Time independent localised solutions

$$w \otimes \eta * f \rightarrow w \otimes f$$

$$q(x) = \int_{\mathbb{R}} dy \, w(x-y) f \circ q(y)$$

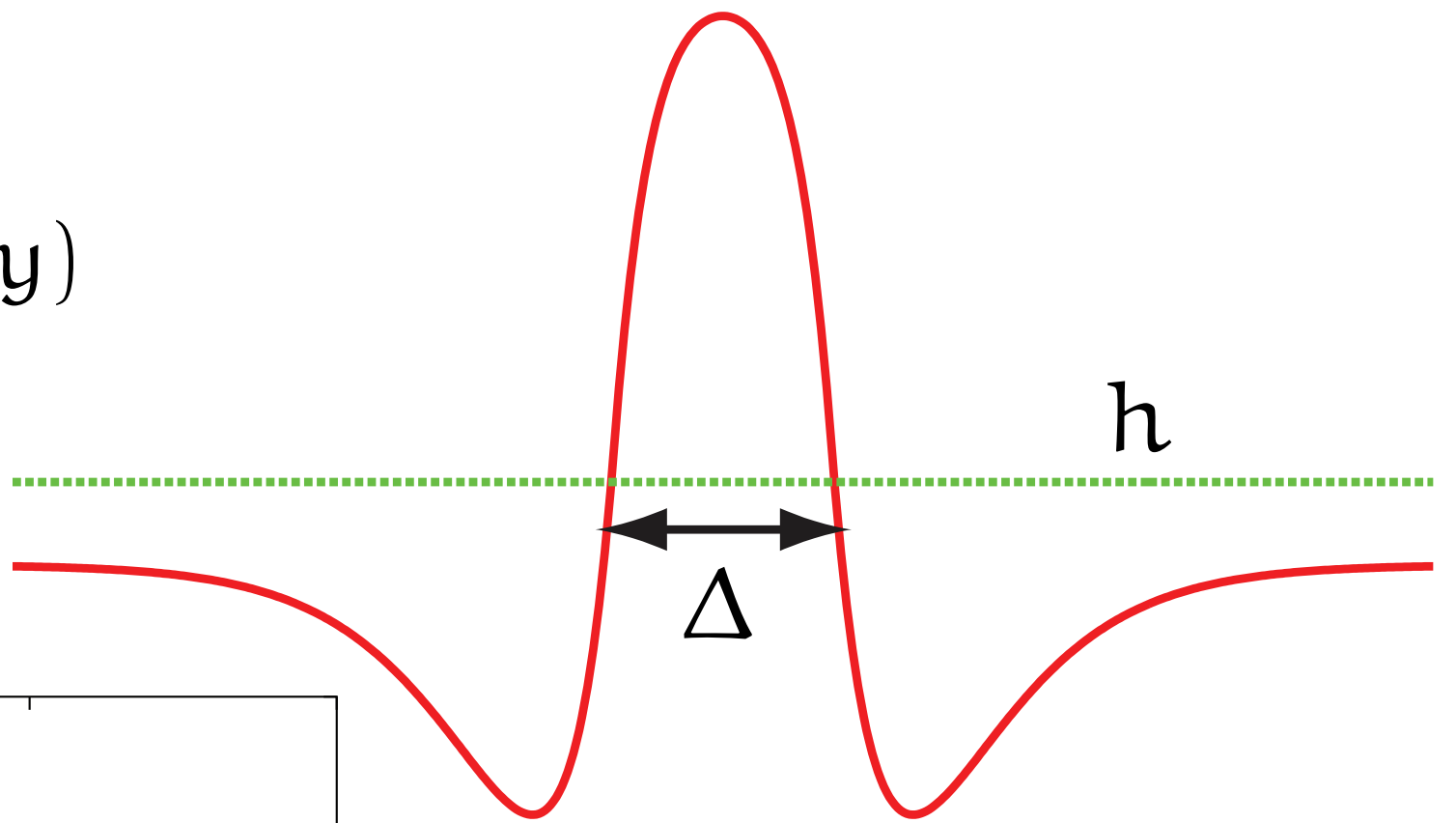


$$w(x) = (1 - |x|)e^{-|x|}$$



Exact result for I-bump: $f(u) = H(u - h)$

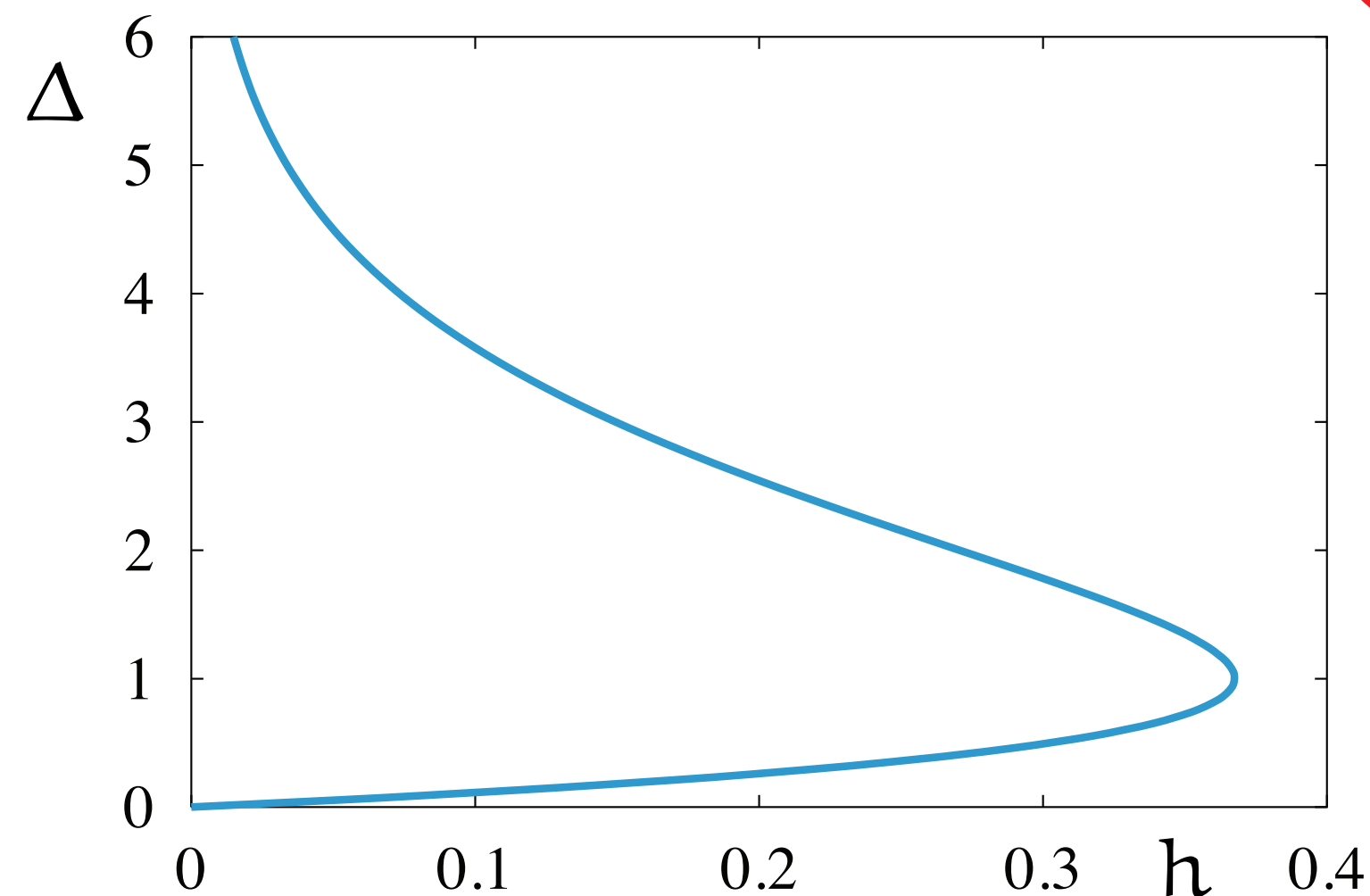
$$q(x) = \int_0^\Delta dy \, w(x - y)$$



$$q(0) = h = q(\Delta)$$

$$\Delta e^{-\Delta} = h$$

working memory



Stability

Examine eigenspectrum of the linearization about a solu

Solutions of form $u(x)e^{\lambda t}$ satisfy $\mathcal{L}u(x) = u(x)$

$$\mathcal{L}u(x) = \tilde{\eta}(\lambda) \int_{-\infty}^{\infty} dy \, w(x-y) f'(q(y) - h) u(y)$$

For Heaviside firing rate

$$f'(q(x)) = \frac{\delta(x)}{|q'(0)|} + \frac{\delta(x - \Delta)}{|q'(\Delta)|}$$

so

$$u(x) = \frac{\tilde{\eta}(\lambda)}{|w(0) - w(\Delta)|} [w(x)u(0) + w(x - \Delta)u(\Delta)]$$

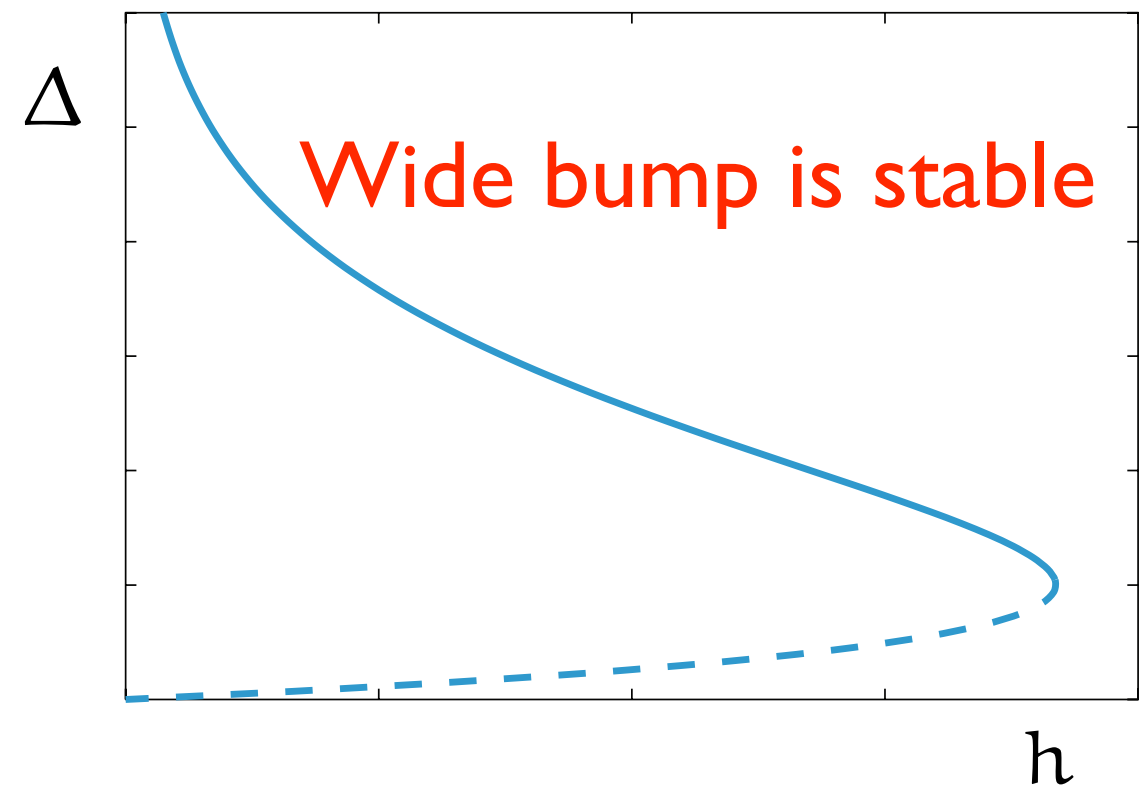
System of linear equations for perturbations at threshold

$$\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix} = \mathcal{A}(\lambda) \begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix}, \quad \mathcal{A}(\lambda) = \frac{\tilde{\eta}(\lambda)}{|w(0) - w(\Delta)|} \begin{bmatrix} w(0) & w(\Delta) \\ w(\Delta) & w(0) \end{bmatrix}$$

Non trivial solution if

$$\mathcal{E}(\lambda) = \det(\mathcal{A}(\lambda) - I) = 0$$

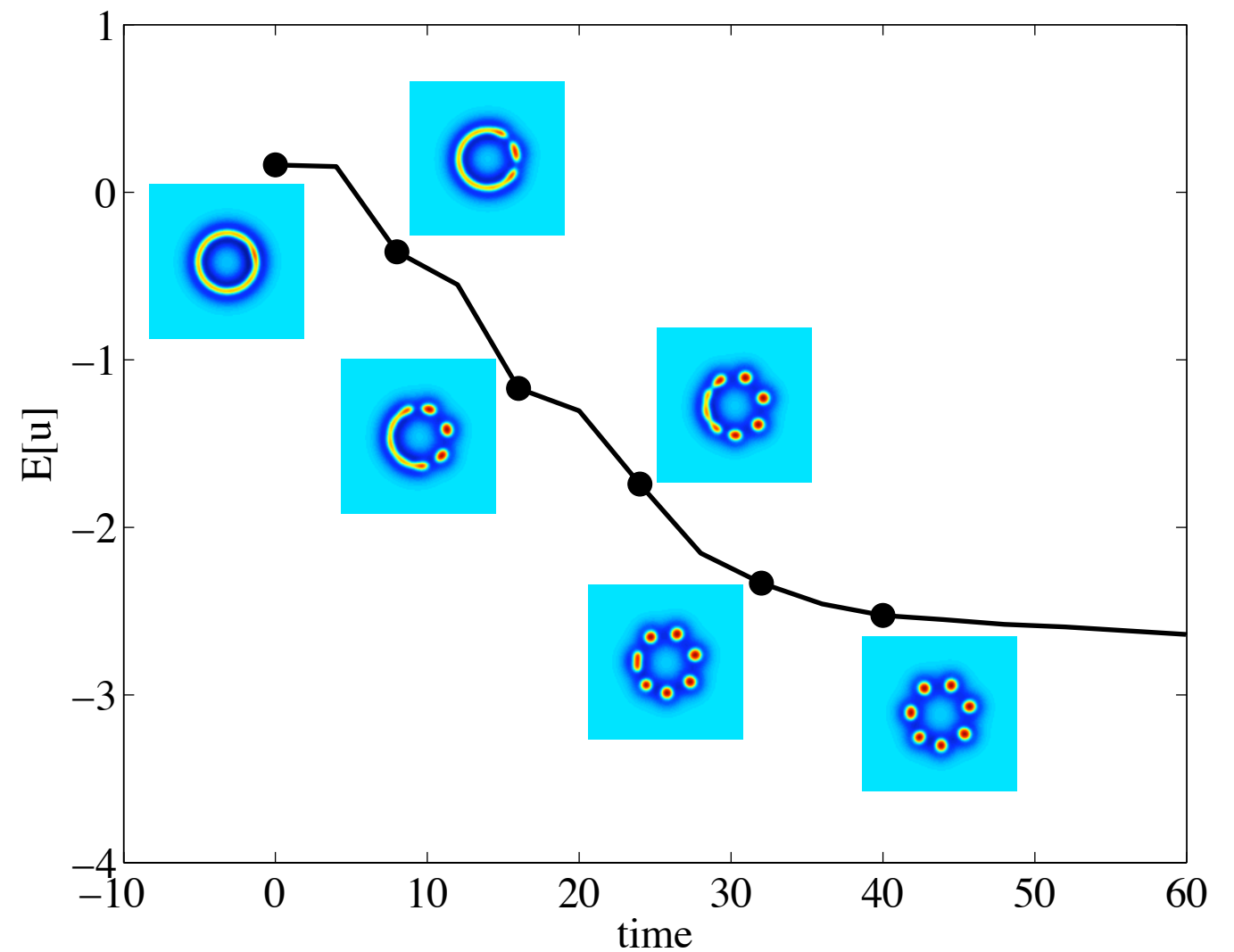
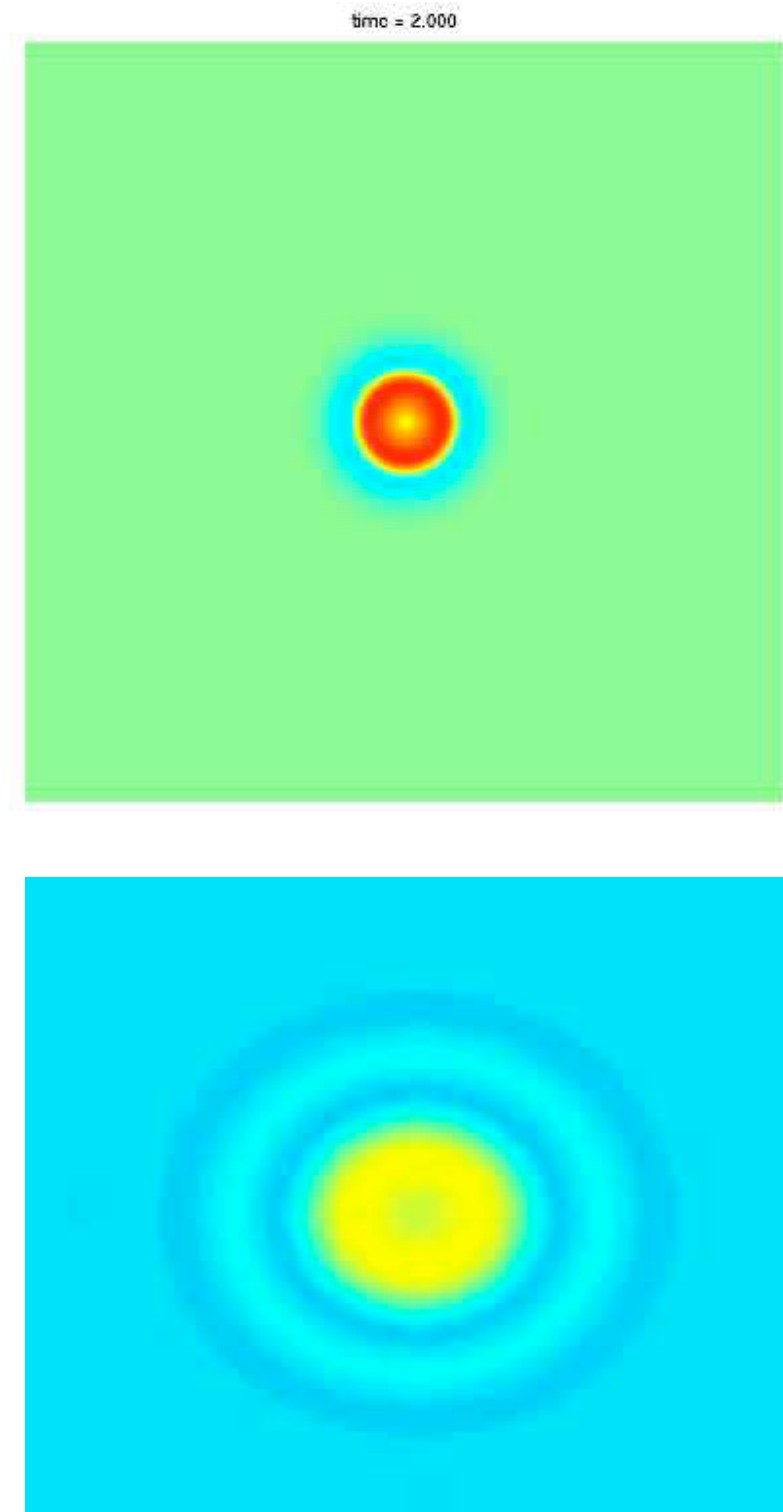
Solutions stable if $\text{Re } \lambda < 0$



Evans function for integral neural field equation

S Coombes and M R Owen (2004) Evans functions for integral neural field equations with Heaviside firing rate function, SIAM Journal on Applied Dynamical Systems, Vol 34, 574-600.

Predictions of Evans function



M R Owen, C R Laing and S Coombes 2007 Bumps and rings in a two-dimensional neural field: splitting and rotational instabilities, New Journal of Physics, Vol 9, 378

Threshold accommodation

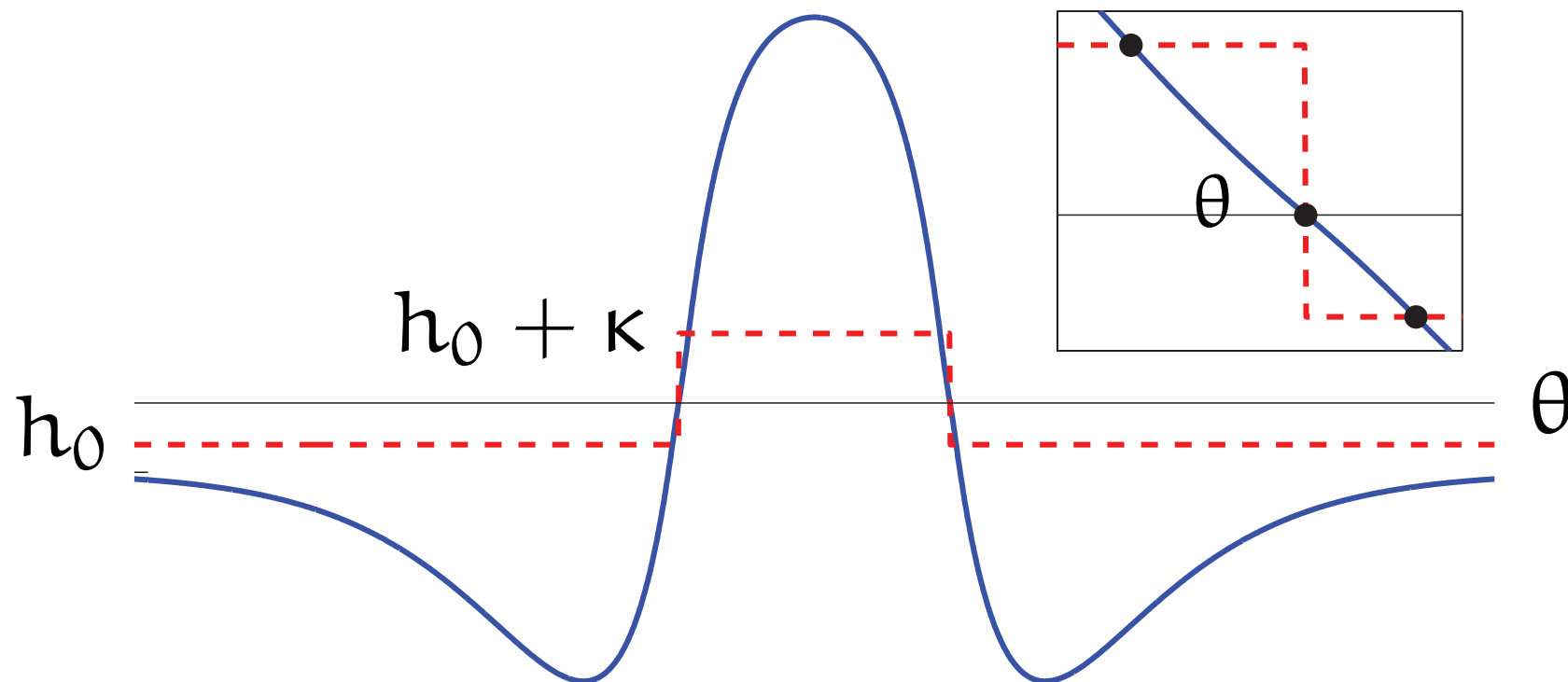
Hill (1936), “... the threshold rises when the *local potential* is maintained ... and reverts gradually to its original value when the nerve is allowed to rest.”

$$\frac{\partial h}{\partial t} = -(h - h_0) + \kappa H(u - \theta)$$

One bump $(u, h) = (q(x), p(x))$

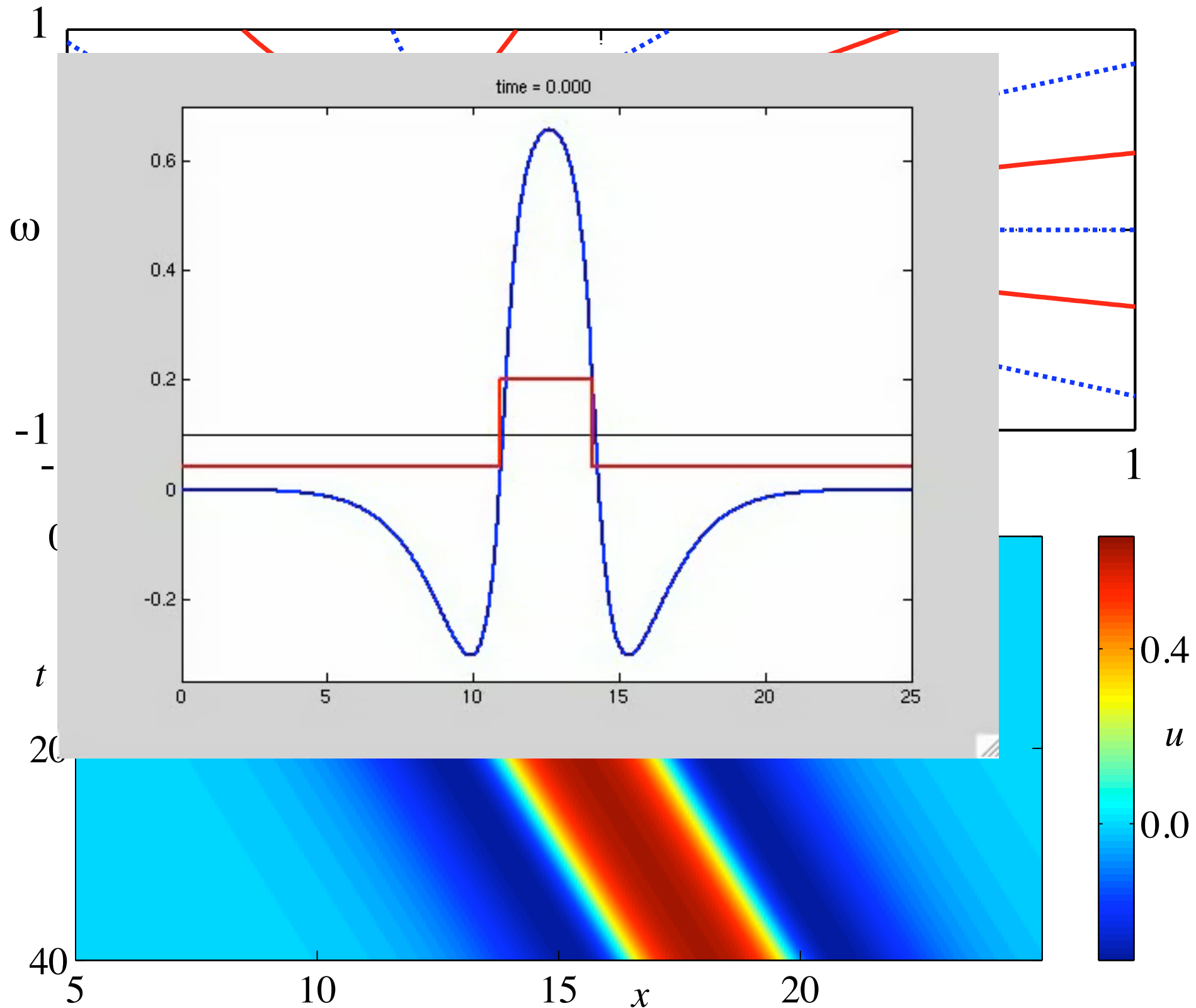
$$q = w \otimes H(q - p)$$

$$p = \begin{cases} h_0 & q < \theta \\ h_0 + \kappa & q \geq \theta \end{cases}$$



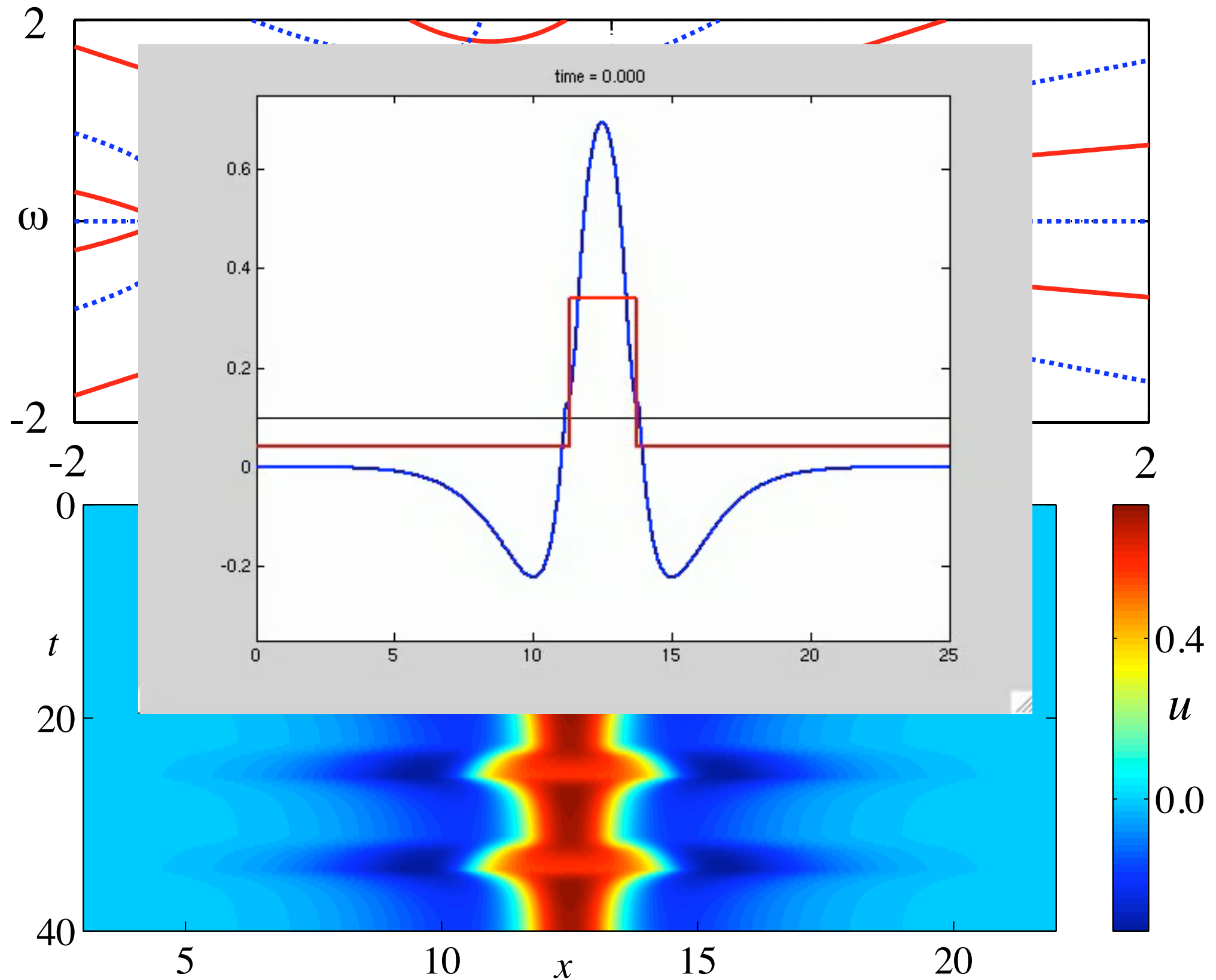
Bump Stability I: $\eta(t) = \alpha^2 t e^{-\alpha t}$

Low κ instability on Re axis (increasing α)

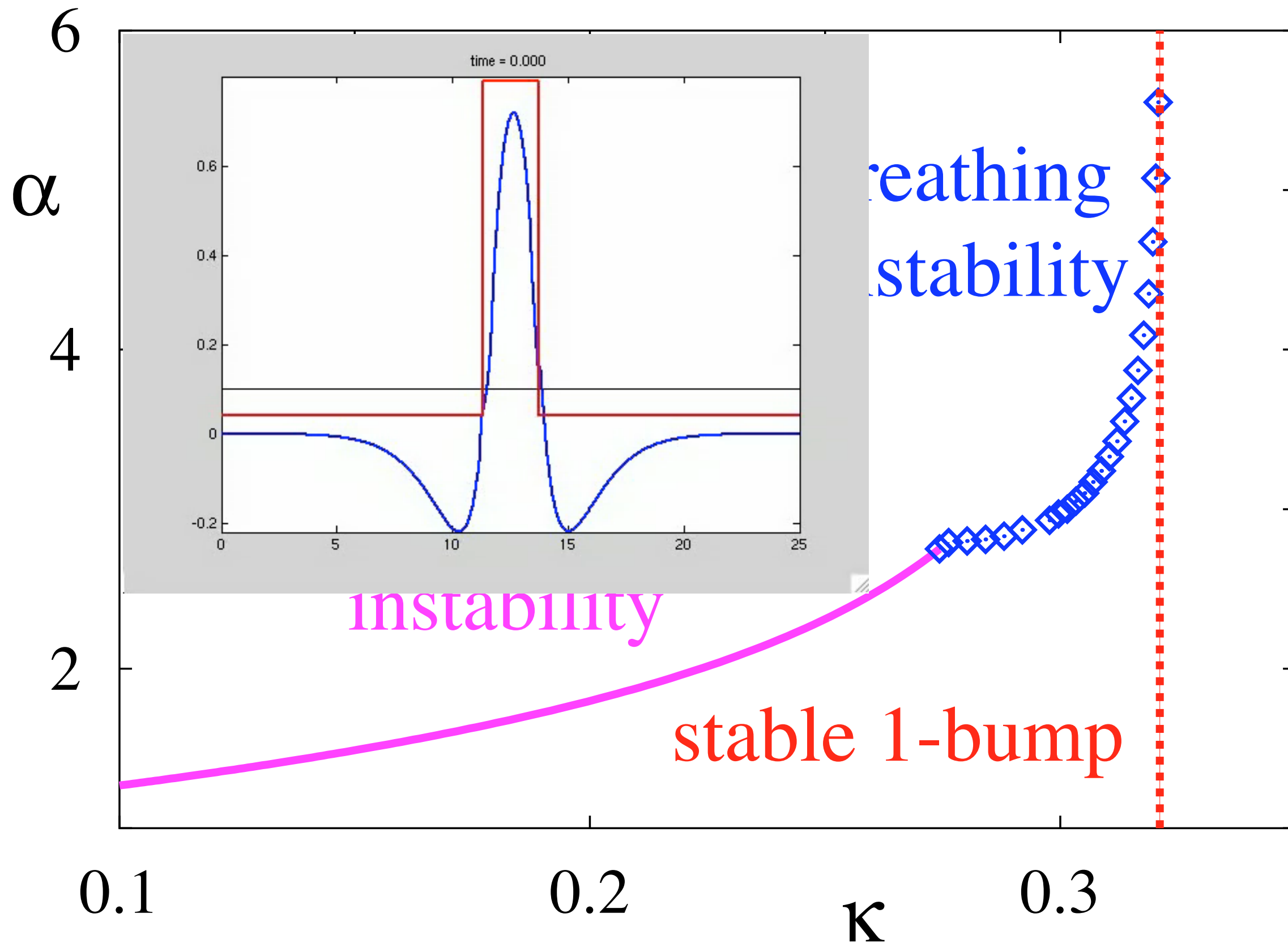


Bump Stability II

High κ instability on Im axis (increasing α) gives a breather

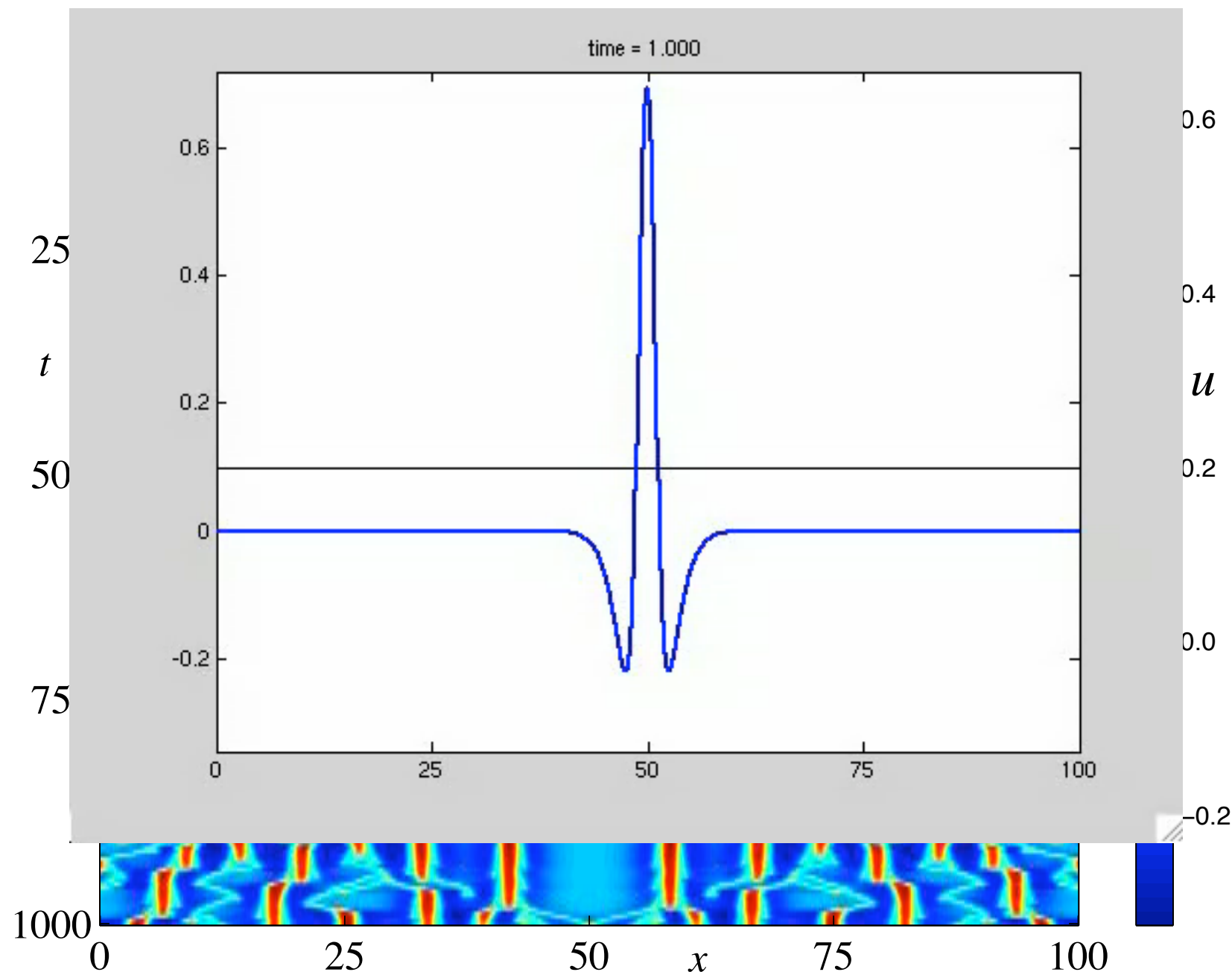


Summary of Bump instabilities



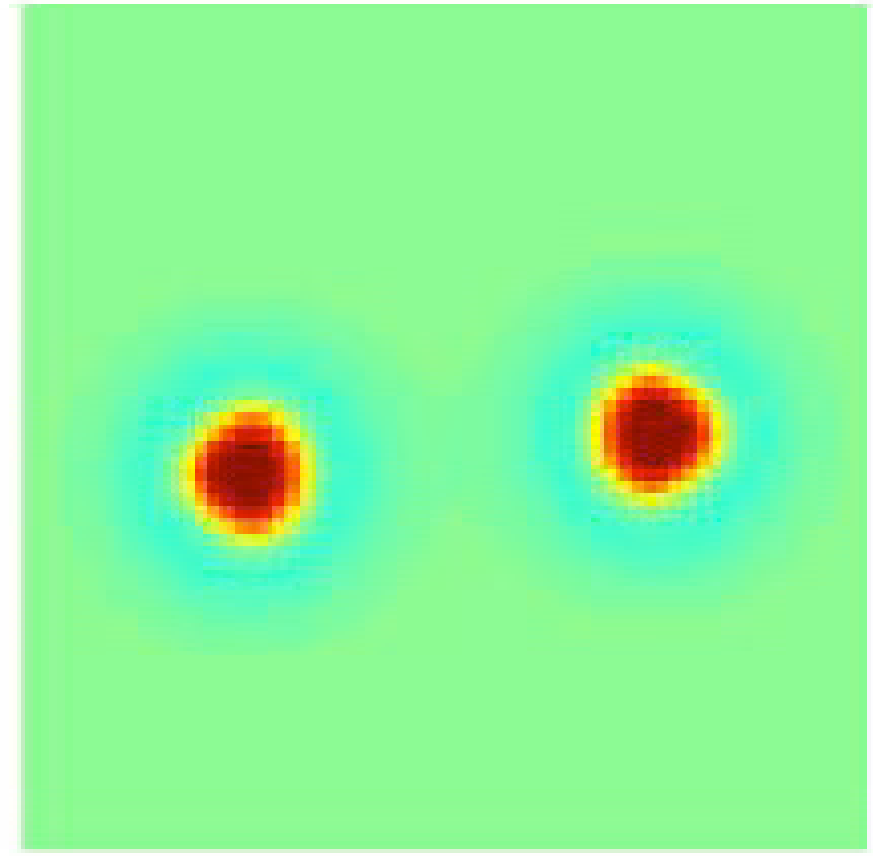
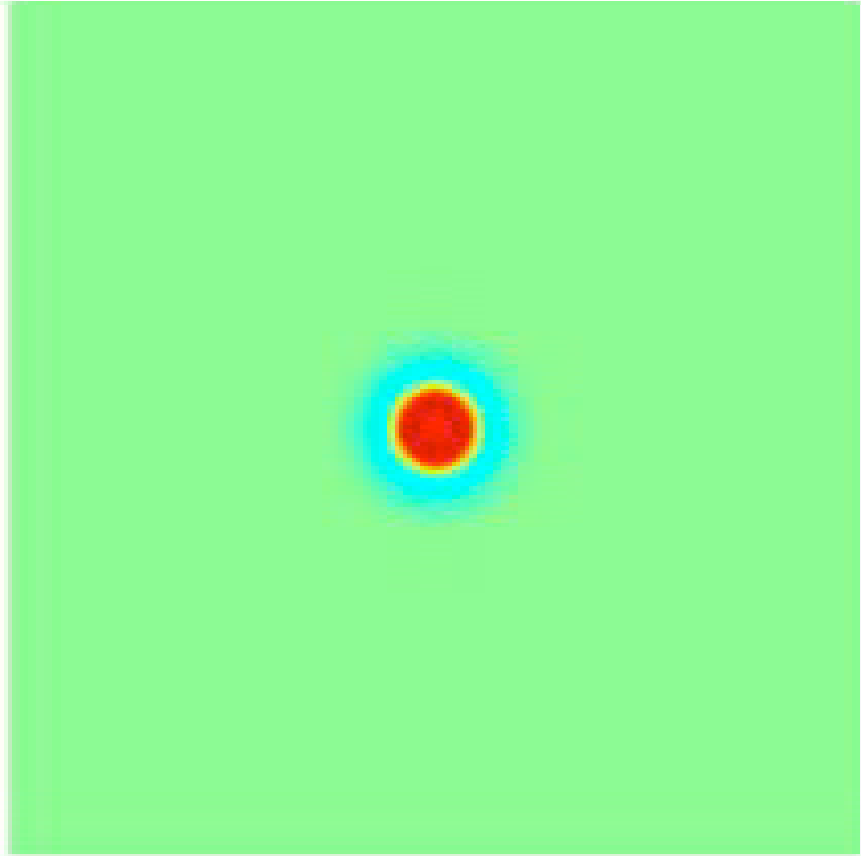
Exotic Dynamics

... including asymmetric breathers, multiple bumps, multiple pulses, periodic traveling waves, and bump-splitting instabilities that appear to lead to spatio-temporal chaos.



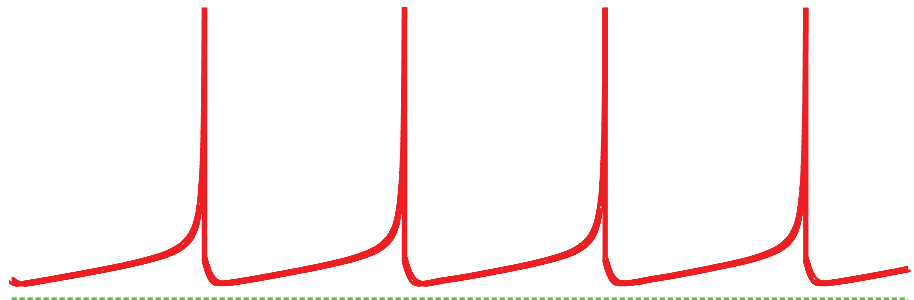
S Coombes and M R Owen: Bumps, breathers and waves in a neural network with spike frequency adaptation. PRL, 94, 148102, (2005).

Splitting and scattering

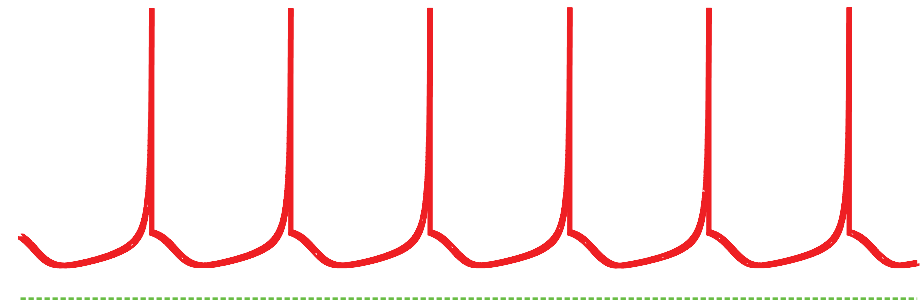


Auto/dispersive solitons as seen in coupled cubic complex Ginzburg-Landau systems and three component reaction-diffusion systems.

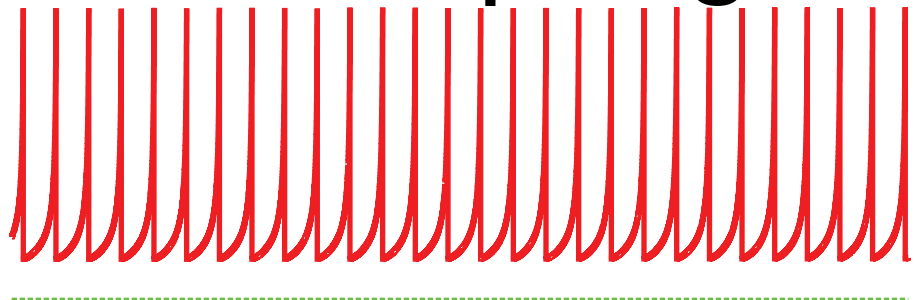
Regular spiking



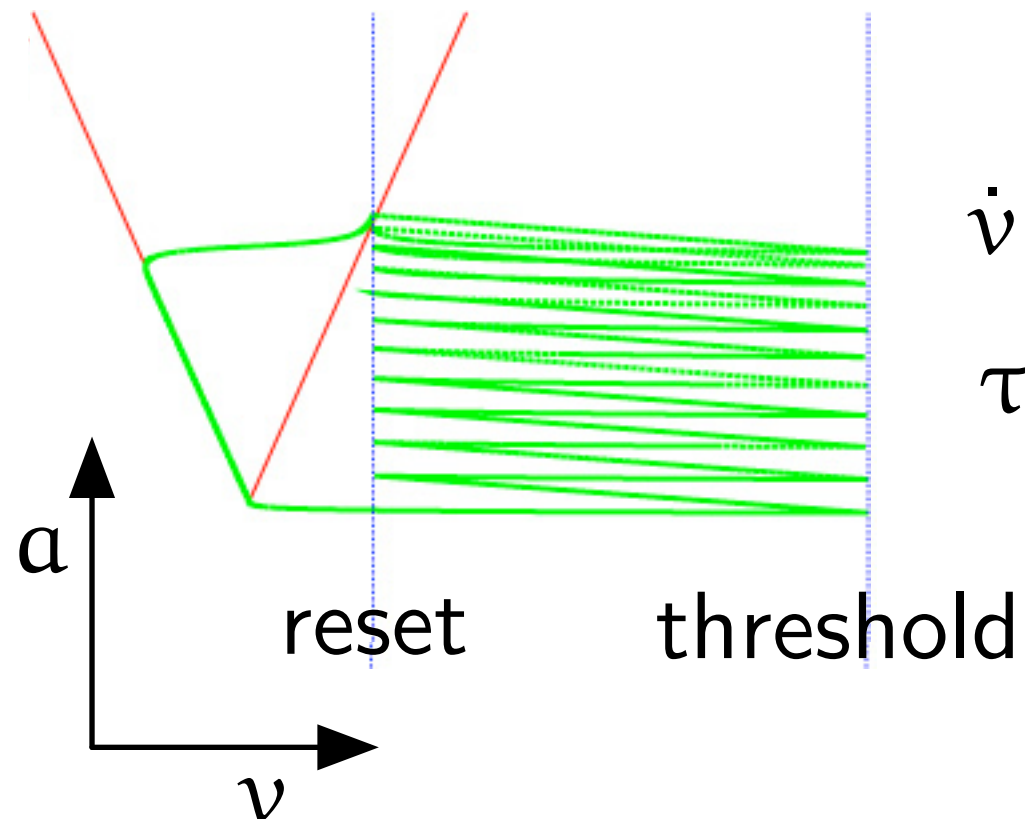
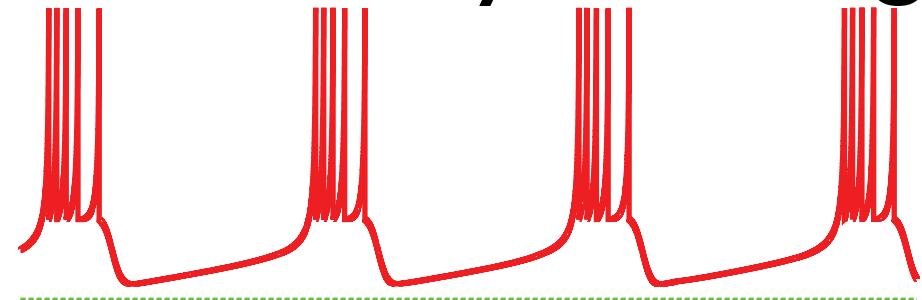
Chattering



Fast spiking

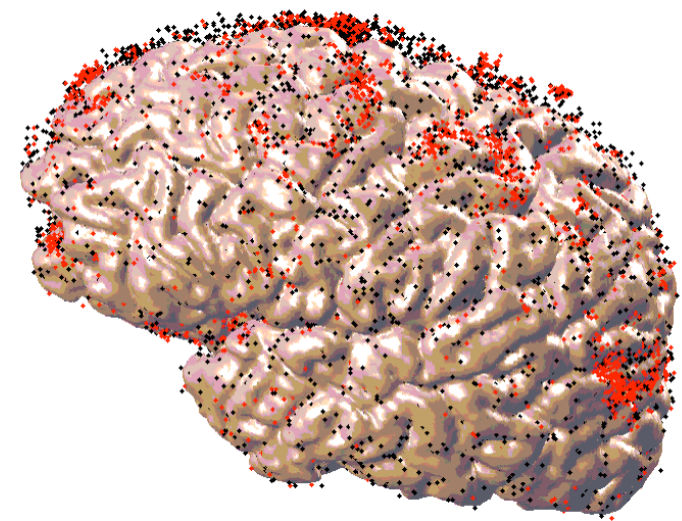


Intrinsically bursting



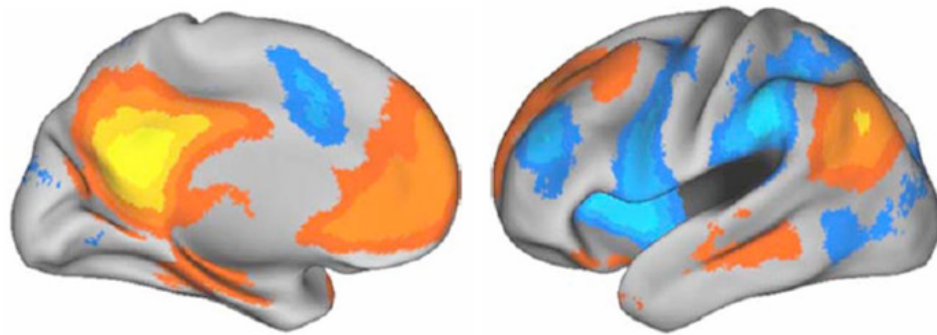
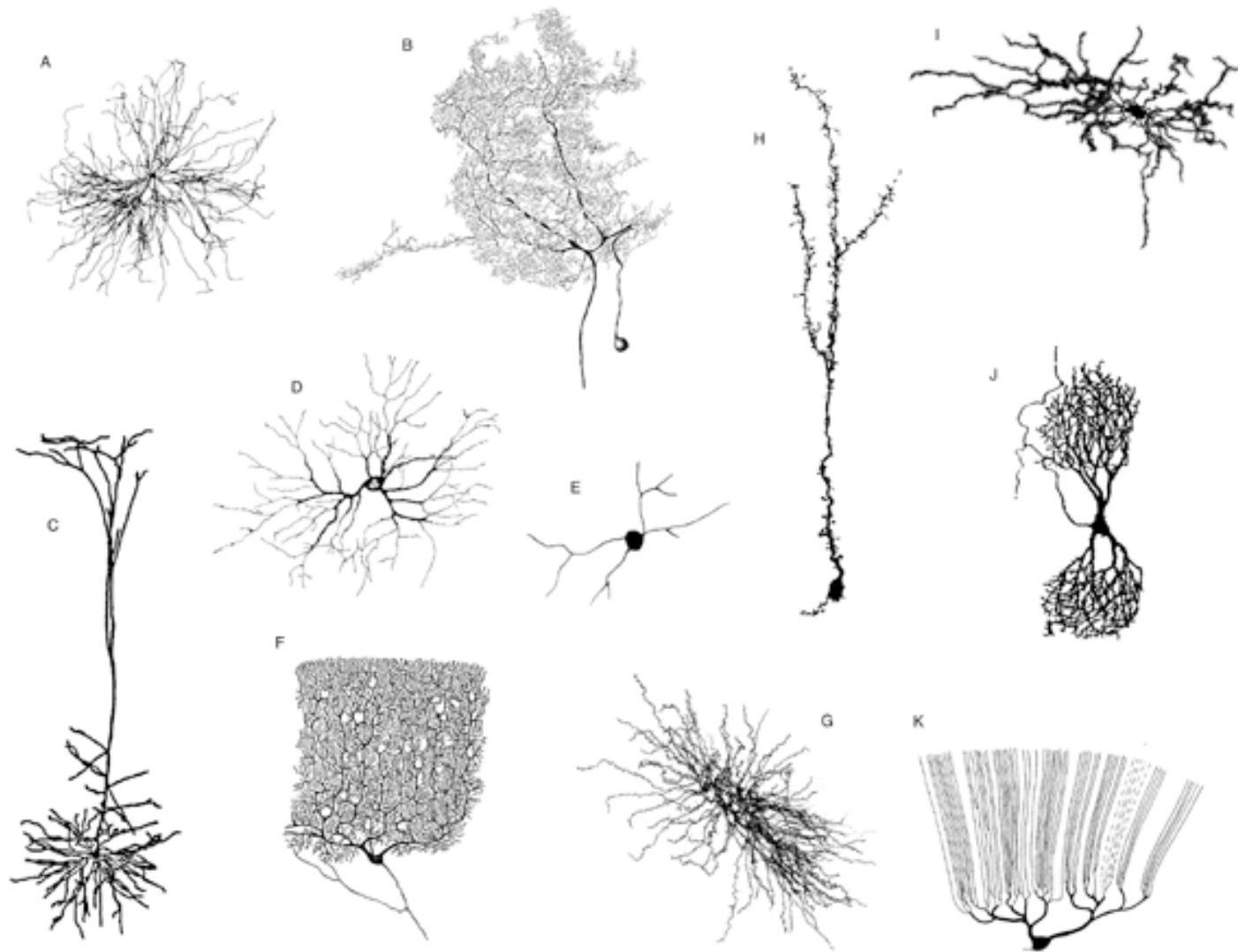
$$\dot{v} = |v| +$$

$$\tau \dot{a} = -a$$

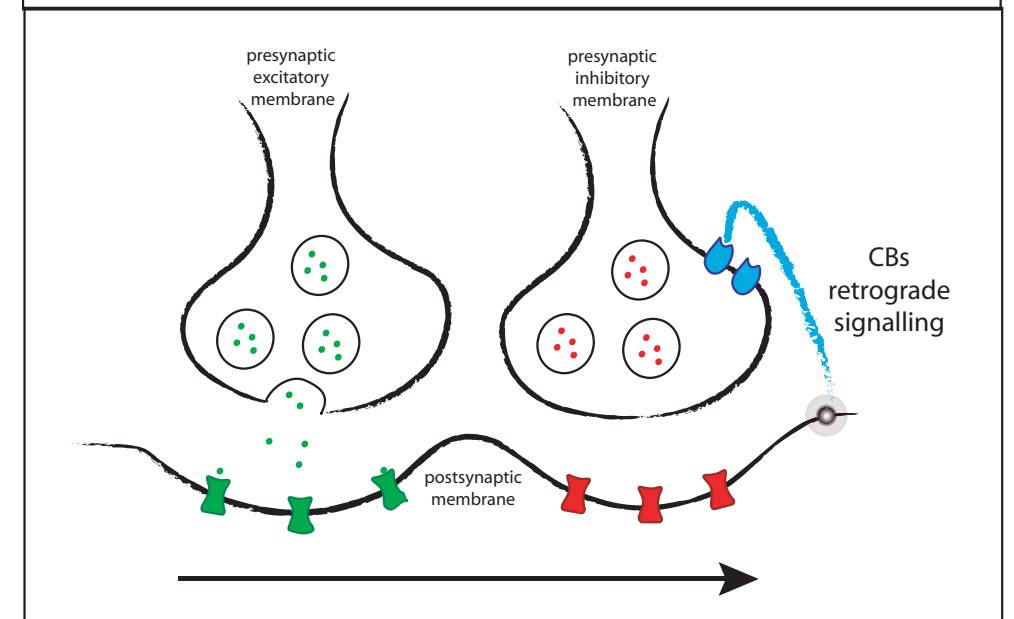
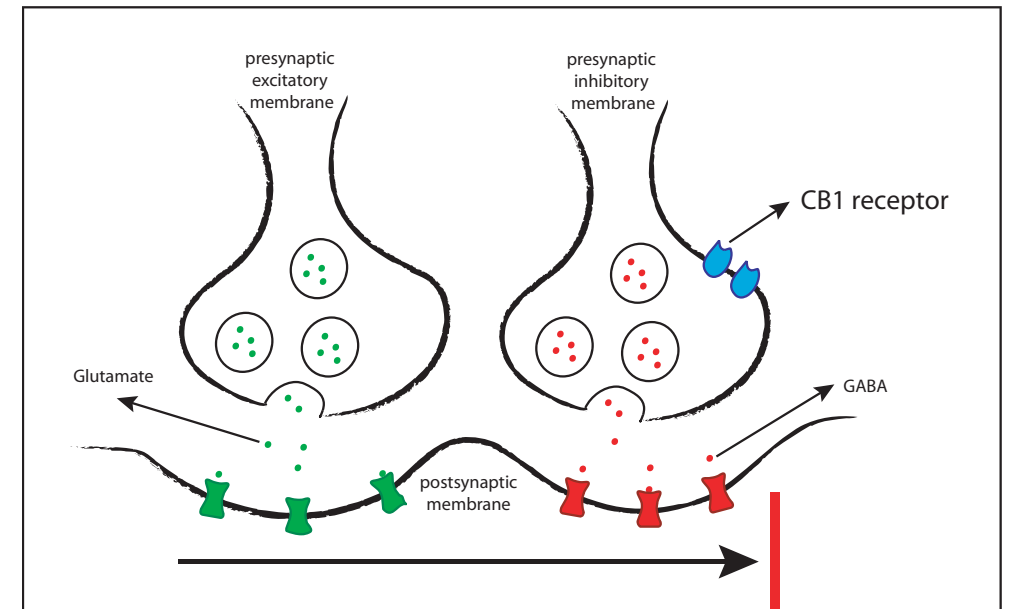
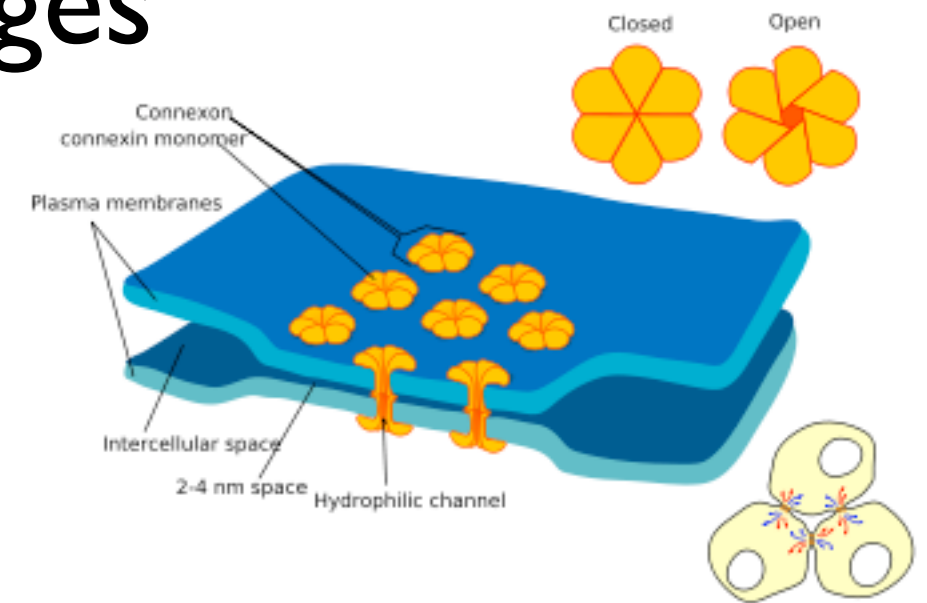


S Coombes and M Zachariou 2009, in
Coherent Behavior in Neuronal Networks
(Ed. Rubin, Josic, Matias, Romo), Springer.

Further Challenges



Default mode network and ultra slow coherent oscillations



In collaboration with

Nikola Venkov
(Notts)



Gabriel Lord
(Heriot-Watt)



Yulia Timofeeva
(Warwick)



David Liley
(Melbourne)

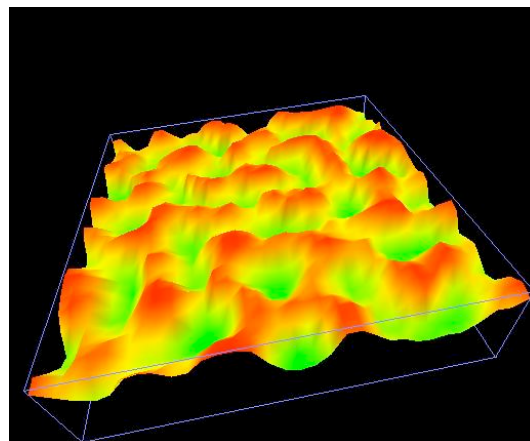


Engineering and Physical Sciences
Research Council

Markus Owen (Notts)



Ingo Bojak (Nijmegen)



Carlo Laing (Massey, NZ)

