Neural Reduction

How can we simulate systems with 1000's of neurons + develop analytic methods to study them?

Firing rate models are one very useful approach

Can we make F.R. models based on spiking models ?

Alex Roxin has shown a couple of approaches

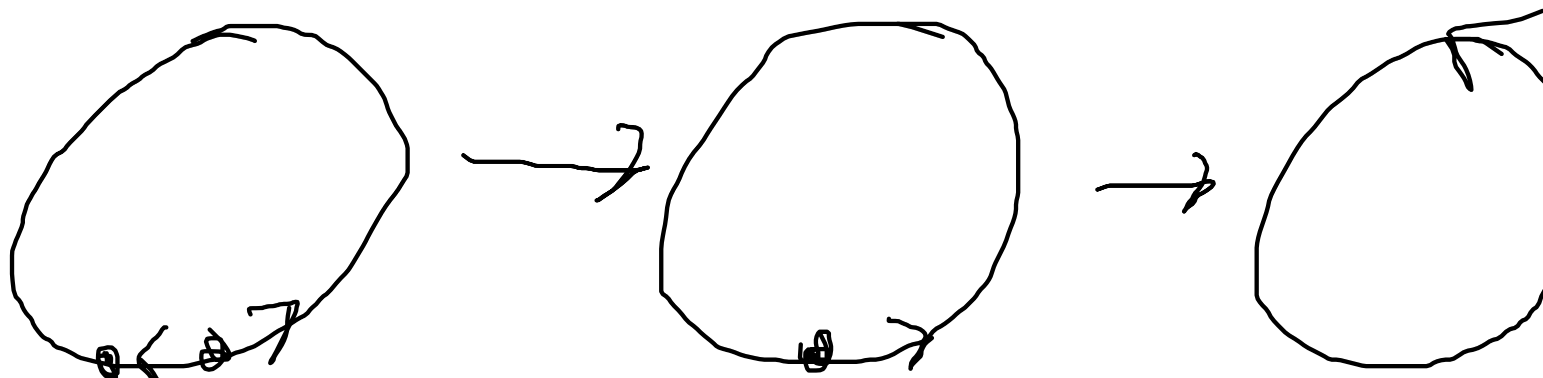
Cowan + others use binary neurons + Markov models

Here, I will use some methods of perturbation and averaging

1. Reductions from bifurcations
2. Slow synapses
3. Noisy FI curve

Reductions from bifurcations

The SNIC bifurcation



$I < I^*$ $\qquad$ $I = I^*$ $\qquad$ $I > I^*$

$\dot V = F(V,n,I)$, $\quad F(\bar V, \bar n, I) = 0$, $\quad A(I) = D_{(V,n)} F(\bar V, \bar n, I)$

$A(I^*)\varphi = 0 \quad A^T(I^*)\psi = 0$, $\quad \varphi \cdot \psi = 1$

Write $(V,n) = (\bar V, \bar n) + \varepsilon x \varphi + \cdots \quad I = I^* + \varepsilon^2 i$

$\dot x = \varepsilon [q x^2 + a i]$ rescale time and x

$$\dot x = x^2 + \alpha i$$

Networks

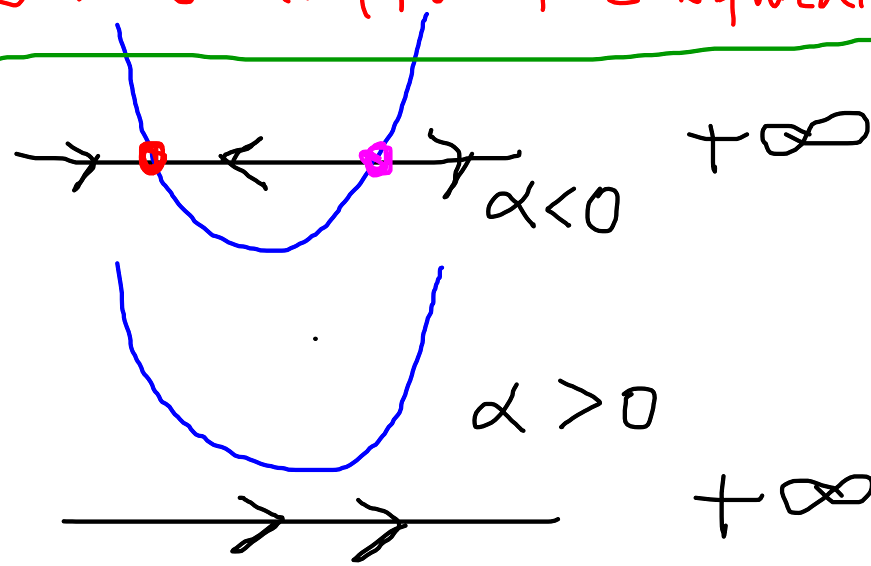
$$\dot V_i = F(V_i, I_i) - \sum \hat g_{ij} s_j (V_i - E_{ij}) \qquad \dot s_i = -s_i/\hat\tau + \delta(V_i - V_T)$$

$$\dot x_i = x_i^2 + \alpha_i + \sum Q_{ij} s_j, \qquad \dot s_j = -s_j/(\hat\tau/\varepsilon) + ??$$

$$Q_{ij} = -\frac{\hat g_{ij}}{\varepsilon^2}(\bar V - E_{ij})$$

ε is distance from the bifurcation!

ASIDE $\dot x = x^2 + \alpha$



When $x \to +\infty$, reset to $x = -\infty$!

QIF. When $x = +\infty$, increment synapses.

To avoid big numbers

$\hat\tau \sim O(\varepsilon)$, $\hat g \sim O(\varepsilon^2)$

Networks, ctd

Recapitulating:

We can qualitatively reduce

$$\dot{V}_i = F(V_i, I_i) - \varepsilon^2 \sum g_{ij} s_j (V_i - E_{ij})$$
$$\dot{s}_i = -\varepsilon s_i/\tau + \delta(V_i - V_T)$$

to

$$x_i' = x_i^2 + \alpha_i + \sum q_{ij} s_j, \quad s_i' = -s_i/\tau + \delta(x - \text{"}\infty\text{"})$$

Same procedure to add slow currents like SFA: $\dot{z}_i = -\varepsilon z_i/\tau_A + \delta(V - V_T)$

$$I_z = g_z (V - E_z)$$

Change of variables to the "theta model"

Let $x_i = \tan \theta_i/2$

$$x_i' = \frac{1}{2\cos^2(\theta_i/2)} \qquad \theta_i' = \frac{\sin^2 \frac{\theta_i}{2}}{\cos^2 \theta_i/2} + \alpha_i + Q_i, \quad s_i' = -s_i/\tau + \delta(\theta_i - \pi)$$

$$\boxed{\theta_i' = 1 - \cos\theta_i + (1 + \cos\theta_i)[\alpha_i + Q_i]}$$

Discussion

- SNIC is global bifurcation, yet admits local desc

- Same methods applied to Hopf bifurcation but, HB is only small amplitude, so relevance to neurons is unclear

$$\dot{z}_i = z_i(\lambda_i - \zeta_i |z_i|^2) + \sum \beta_{ij} z_j$$

(See AEK) complex parameters

- Closely related to Izhikevich model

$x' = f(x) - y$, $y' = a(bx - y)$ $f(x)$ is quadratic

when $x = x_T$, $x \to x_R$, $y \to y + c$

$b = 0$, $x_T = +\infty$, $x_R = -\infty$

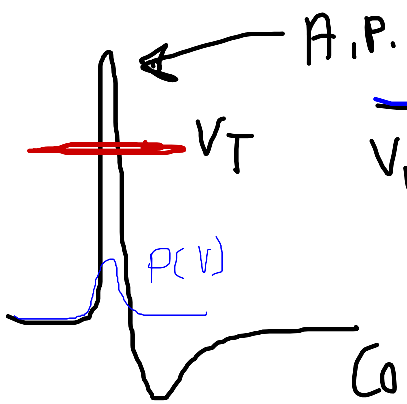
- Analysis is harder than LIF, but can be numerically solved pretty easily

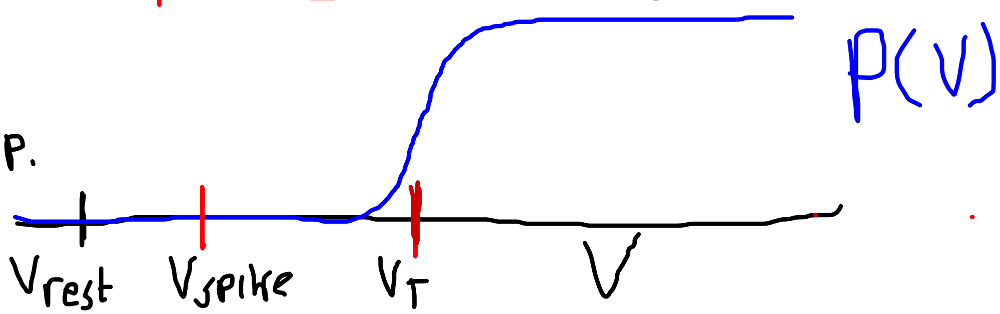
Firing rate models

Slow synapses

$\dot{V}_1 = F(V_1, g_2)$, $\dot{V}_2 = F(V_2, g_1)$ $\qquad g = \sum g_{ij} S_j$ input conductances

$\dot{S}_i = \varepsilon[-S_i + p(V_i)]$ slow synapse!



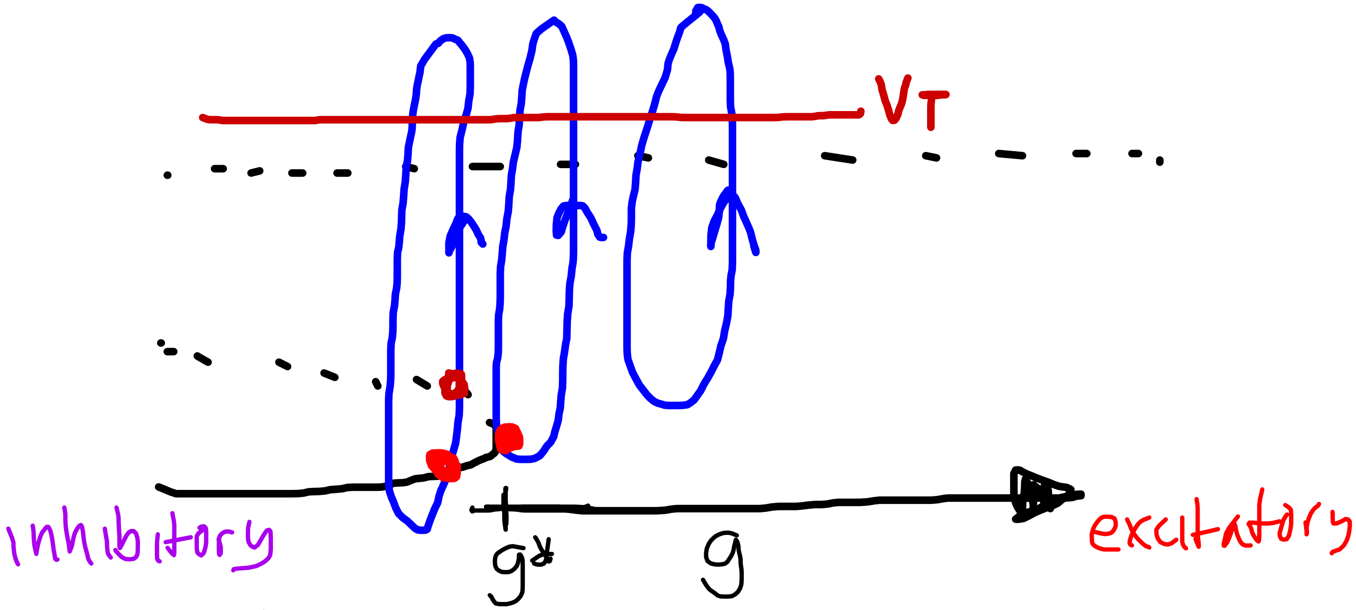


Since S_i are "slow", hold them constant

Consider first, excitatory synapses

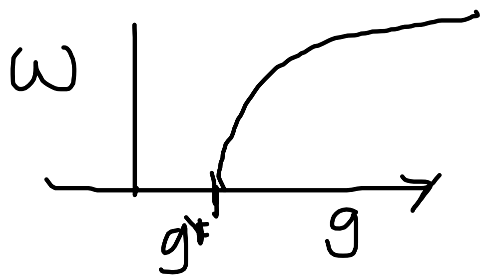
As $S \nearrow$, post syn cell begins to fire ASSUME via SNIC

Slow synapses, ctd



For $g<g^*$ $V(t;g) \to$ fixed point near rest

$g>g^*$ $V(t;g) \to$ SLC with period $T(g)$, $\omega(g)$ frequency



$$\omega \sim K\sqrt{g-g^*}$$

Averaging

$\dot{s} = \varepsilon[-s + p(V(t;g))]$ $g > g^*$ $V(t;g)$ oscillates

$$\dot{s} \approx \varepsilon\left[-s + \frac{1}{T(g)}\int_0^{T(g)} p(V(t;g))\,dt\right]$$

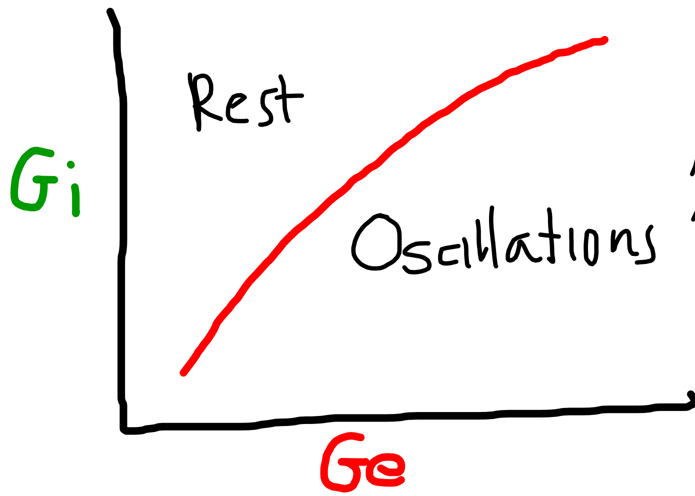
If the width of the AP is roughly independent of the frequency, a reasonable assumption then

$$\dot{s} \approx \varepsilon[-s + \omega(g)\gamma]$$

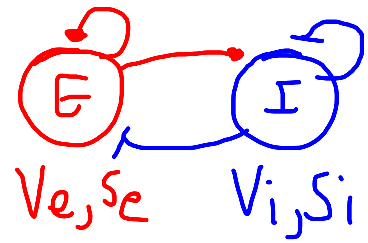
that is: $s(t)$ relaxes to the firing rate ω!

E+I coupling

$$I_{syn} = G_e(t)(V - E_{ex}) + G_i(t)(V - E_{in})$$



$$\approx \quad C_e G_e - C_i G_i - \theta_e = 0$$



$$\dot{S}_e = \varepsilon_e\left(-S_e + \beta_e W_e\left(C_{ee} S_e - C_{ie} S_i - \theta_e\right)\right)$$

$$\dot{S}_i = \varepsilon_i\left[-S_i + \beta_i W_i\left(C_{ei} S_e - C_{ii} S_i - \theta_i\right)\right)$$

WC EQNS!

We have reduced a pair of coupled spiking models to a pair of equations for their synaptic activity driven by the F-I curve

Related methods

LIF + noise + slowly varying current

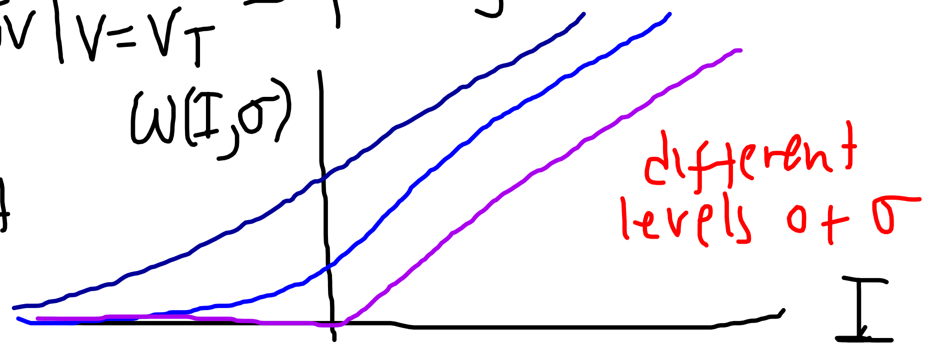
$$\dot{V} = I(t) - V + \sigma \xi$$

Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial V}\left[P[I - V] \right] + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial V^2} + \delta(V - V_{reset})\, J(t)$$

$$J(t) = -\frac{\sigma^2}{2}\frac{\partial P}{\partial V}\bigg|_{V=V_T} \equiv \text{firing rate}$$

If I is constant



$$\tau_? \frac{d\nu}{dt} = -\nu + \omega(I(t);\sigma)$$

A "firing rate" model but what is $\tau_?$ ☆ !