Dynamical modelling of systems of coupled oscillators

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Peter Ashwin

University of Exeter

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The anatomy of an oscillator

Q: What is an oscillator?
A: A dynamical system that produces periodic behaviour.

For example, in $\mathbf{R}^d$:

$$\dot{x}_1 = f_1(x_1, \cdots, x_d), \quad \dot{x}_d = f_d(x_1, \cdots, x_d)$$

with a periodic orbit

$$P(t) = (p_1(t), \cdots, p_d(t))$$

with period $T > 0$, i.e.

$$P(t+T) = P(t)$$

such that T is smallest possible choice of periodicity of all components.
We consider *stable limit cycle oscillators* of ODEs for any initial condition x that starts close enough to $P(t)$ in all components we have

$$|x(t) - P(t+\phi)| \to 0$$

as $t \to \infty$ for some ϕ.

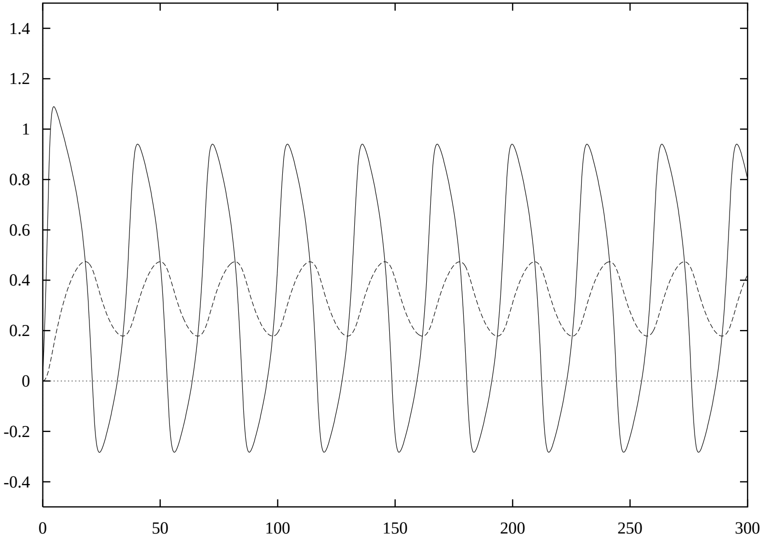
Motivating example

Consider the Fitzhugh-Nagumo system

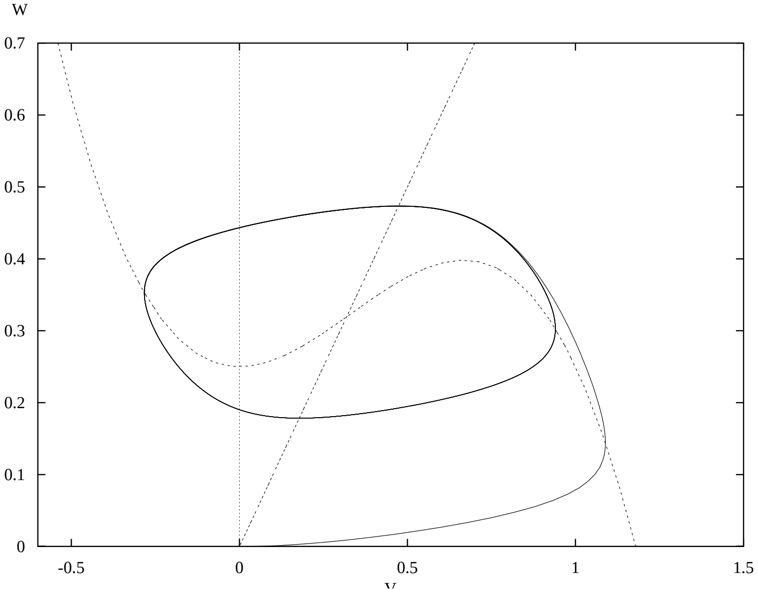
$$\begin{aligned}\dot{V} &= F(V) - W + I \\ \dot{W} &= \epsilon(V - \gamma W)\end{aligned}$$

with $F(V) = V(1 - V)(V - A)$ and parameters

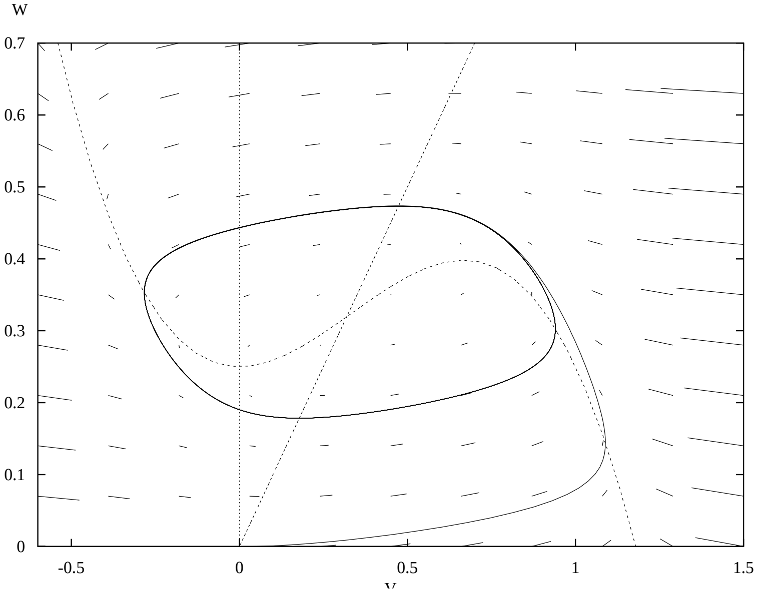
$$A = .25,\ \epsilon = .05,\ \gamma = 1,\ I = .25$$



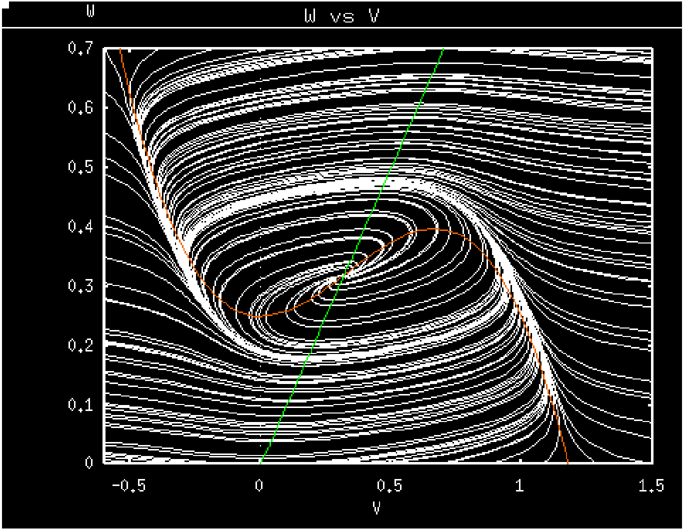
Time-series of typical solution.



Phase plane of typical solution.



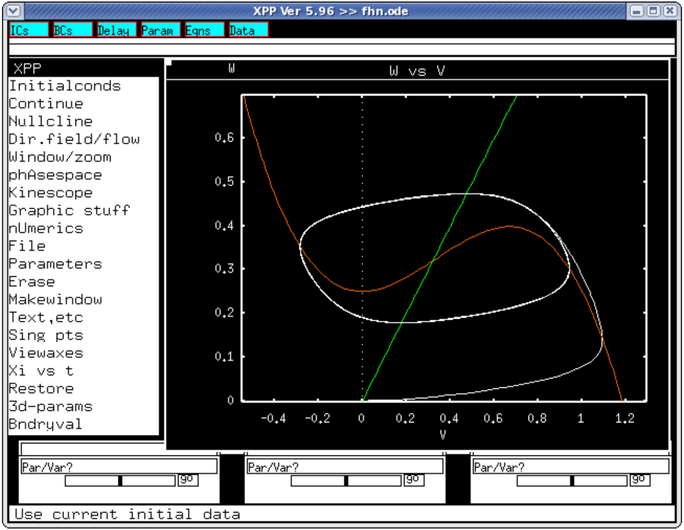
Phase plane of typical solution with directions of flow.



Phase plane of typical solution with flow added.

Code for xppaut [Ermentrout]:
http://www.math.pitt.edu/~bard/xpp/xpp.html

```
dv/dt = f(v)-w+s(t)+I_0
dw/dt = eps*(v-gamma*w)
f(v)=v*(1-v)*(v-a)
s(t)=al*sin(omega*t)
param a=.25,eps=.05,gamma=1,I_0=.25
param al=0,omega=2
@ total=100,dt=.2,xhi=100
done
```


XPP Ver 5.96 >> fhn.ode
ICs
BCs
Delay
Param
Eqns
Data
XPP
Initialconds
Continue
Nullcline
Dir.field/flow
Window/zoom
phAsespace
Kinescope
Graphic stuff
nUmerics
File
Parameters
Erase
Makewindow
Text,etc
Sing pts
Viewaxes
Xi vs t
Restore
3d-params
Bndryval
W
W vs V
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
V
Par/Var?
go
Par/Var?
go
Par/Var?
go
Use current initial data

Give it a try?

Different types of periodic oscillations:

- Weakly nonlinear oscillations, e.g. Near-onset small amplitude oscillations, Hopf bifurcation.
- Relaxation oscillations, e.g. Fitzhugh Nagumo, Hodgkin-Huxley models
- Hybrid/switched system oscillations, e.g. leaky integrate-and-fire models

In all cases can be modelled as a *phase oscillator*

$$\dot{\theta} = \omega$$

for θ modulo 2π when transients have decayed, where frequency

$$\omega = \frac{2\pi}{T}$$

related to the period T. Off the limit cycle, however, other dynamics are at work.

Isochrons

Suppose $X \in \mathbf{R}^d$ with

$$\dot{X} = F(X)$$

has a stable limit cycle $P(t)$, period T. We define the set of points with eventual phase ϕ to be

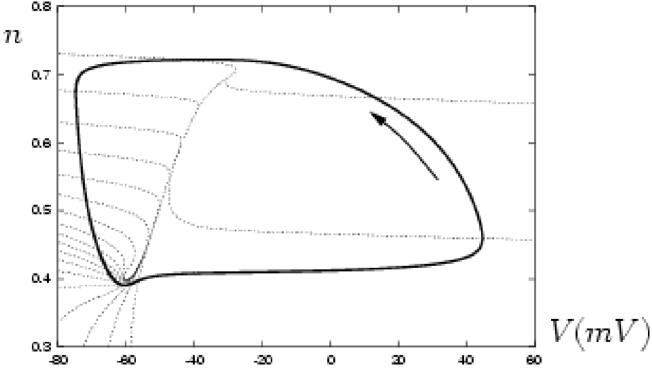
$$I_\phi = \{Y \in \mathbf{R}^d \ : \ |Y(t) - P(t+\phi)| \to \infty\}$$

The sets I_ϕ are called the *isocrons* of the limit cycle. For a stable limit cycle:

- They are manifolds of dimension $d-1$.
- They foliate a neighbourhood of the cycle.
- They can be used to understand the behaviour of forced or coupled oscillators.

K. Josic, E. Brown, J. Moehlis:
http://www.scholarpedia.org/article/Isochron

E Izhikevich:
http://www.izhikevich.com



Isocrons for a Hodgkin-Huxley neuron
(http://www.scholarpedia.org/article/Isochron)

Phase response curves

The *phase response curve* is a way of measuring the response to sudden change in one variable; if we consider a vector perturbation $Z \in \mathbf{R}^d$ then

$$PRC(\theta) = \{\phi : I_\phi \text{ contains } P(\theta) + Z\}.$$

Equivalently, starting at $P(\theta)$ we impulsively change to

$$X(0) = P(\theta) + Z$$

and allow the system to evolve forwards in time. We choose PRC so that

$$|X(t) - P(t + PRC(\theta))| \to 0$$

as $t \to \infty$.

The phase response curve models the change in phase exactly, even for large perturbations if a long enough settling time between perturbations is allowed.

Can apply to continuous perturbations using various equivalent approaches to obtain the *infinitesimal phase response curve*.

Suppose that

$$\dot{X} = F(X) + \epsilon G(t)$$

where $G(t)$ represents forcing and F has an attracting limit cycle $P(t)$.

Assume that unperturbed oscillator $P(t)$ has period 2π.

Kuramoto's approach: We define phase $\Theta(X)$ of all points $X \in \mathbf{R}^d$ by using the isochrons I_ϕ:

$$\phi = \Theta(X) \;\Leftrightarrow\; X \in I_\phi.$$

Note that for the system with $\epsilon = 0$ we have

$$\frac{d}{dt}\left[\Theta(X(t))\right] = \nabla\Theta \cdot \frac{dX}{dt} = \nabla\Theta . F(X)$$

But $\dot{\phi} = 1$ so

$$\nabla\Theta \cdot F(X) = 1.$$

Hence for the case $\epsilon \neq 0$ we have

$$\frac{d\phi}{dt} = 1 + \epsilon\nabla\Theta \cdot G(t).$$

Adjoint approach (Malkin): Note that if the unperturbed oscillators is linearly stable, then the perturbed equation is to first order in ϵ given by

$$\dot{\theta} = 1 + \epsilon Q(\theta) \cdot G(t)$$

where $Q(t)$ is the solution to the *adjoint variational equation*

$$\dot{Q} = -\{DF(P(t))\}^T Q, \quad \text{such that } Q(0) \cdot F(P(0)) = 1.$$

Note that

$$\begin{aligned} \frac{d}{dt}(Q \cdot F) &= \dot{Q} \cdot F + Q \cdot \dot{F} \\ &= -(DF)^T Q \cdot F + Q \cdot (DF)F = 0. \end{aligned}$$

Hence the solutions of the AVE satisfy

$$Q(t) \cdot F(P(t)) = 1$$

for all t.

Coupled phase dynamics

Consider two symmetrically coupled oscillators

$$\begin{aligned} \frac{dX_1}{dt} &= F(X_1) + \epsilon G_1(X_2, X_1) \\ \frac{dX_2}{dt} &= F(X_2) + \epsilon G_2(X_1, X_2) \end{aligned}$$

for ϵ small (weak coupling).
In terms of phases we have approximately

$$\begin{aligned} \frac{d\theta_1}{dt} &= 1 + \epsilon Q(\theta_1) \cdot G_1(P(\theta_2), P(\theta_1)) \\ \frac{d\theta_2}{dt} &= 1 + \epsilon Q(\theta_2) \cdot G_2(P(\theta_1), P(\theta_2)). \end{aligned}$$

Because the phase difference evolves on a slower timescale that the phases, we get an interaction function that expresses the effect of X_2 on X_1 that can be written

$$H_1(\theta) = \frac{1}{T}\int_0^T Q(t)G_1(X_0(t+\theta), X_0(t))\,dt$$

where Q is the solution of the adjoint variational equation.
Method of *averaging* allows us to write previous equation (to $O(\epsilon^2)$) as

$$\begin{aligned}\theta_1' &= 1+\epsilon H_1(\theta_2-\theta_1)\\ \theta_2' &= 1+\epsilon H_2(\theta_1-\theta_2).\end{aligned}$$

For two identically coupled oscillators we set $\phi = \theta_2 - \theta_1$ and obtain

$$\dot{\phi} = -\epsilon(H_1(\phi) - H_2(-\phi)) = \epsilon g(\phi)$$

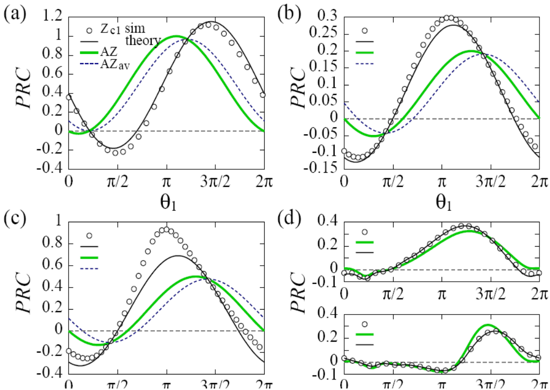
Similarly, starting at a set of N weakly coupled identical phase oscillators $\theta_1, \cdots, \theta_N$, we can reduce to a set of $N-1$ phase differences

$$\phi_i = \theta_i - \theta_N$$

and can obtain

$$\dot{\phi}_i = \epsilon g_i(\phi_1, \cdots, \phi_{N-1})$$

with evolution on a slow timescale.



Example of $g(\theta)$ for various coupling functions. (d) are for gap junction-coupled Morris-Lecar Neurons (from T.-W. Ko and G.B. Ermentrout, arXiv:0809.3371v1 2008)

Synchrony and coupling

Now consider N coupled oscillators reduced to phases:

$$\dot{\theta}_i = \omega + G(\theta_1 - \theta_i, \cdots, \theta_N - \theta_i).$$

Simple model with "additive coupling" is

$$\dot{\theta}_i = \omega + \sum_{j \neq i} K_{ij} g(\theta_i - \theta_j)$$

with K_{ij} coupling strengths; $K_{ij} = K$ for global (mean field) coupling.

Simple choices for phase response curve $g(\phi)$:

- Kuramoto

$$g(\phi) = -\sin(\phi)$$

- Kuramoto-Sakaguchi

$$g(\phi) = -\sin(\phi + \alpha)$$

- Hansel-Mato-Meunier

$$g(\phi) = -\sin(\phi + \alpha) + r\sin(2\phi)$$

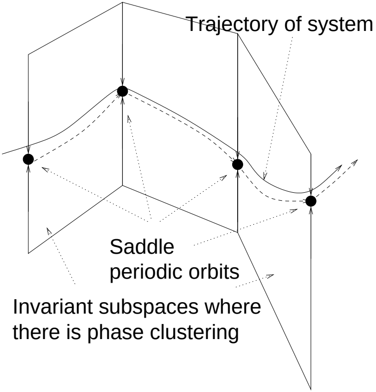
More general: Daido et al:

$$g(\phi) = \sum_n (a_n \cos n\phi + b_n \sin n\phi)$$

Cluster switching/ slow oscillations

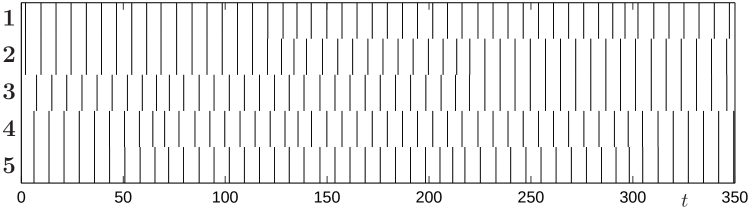
Can find open regions in parameter space (for all $N \geq 4$) where the only attractors consist of robust heteroclinic networks made up of:

- Periodic orbits with nontrivial clustering.
- Unstable manifolds of these periodic orbits.
- *Winnerless competition* between cluster states (Afraimovich, Huerta, Laurent, Nowotny, Rabinovich et al)
- *Slow oscillations/switching dynamics* (Hansel et al, Kori and Kuramoto)

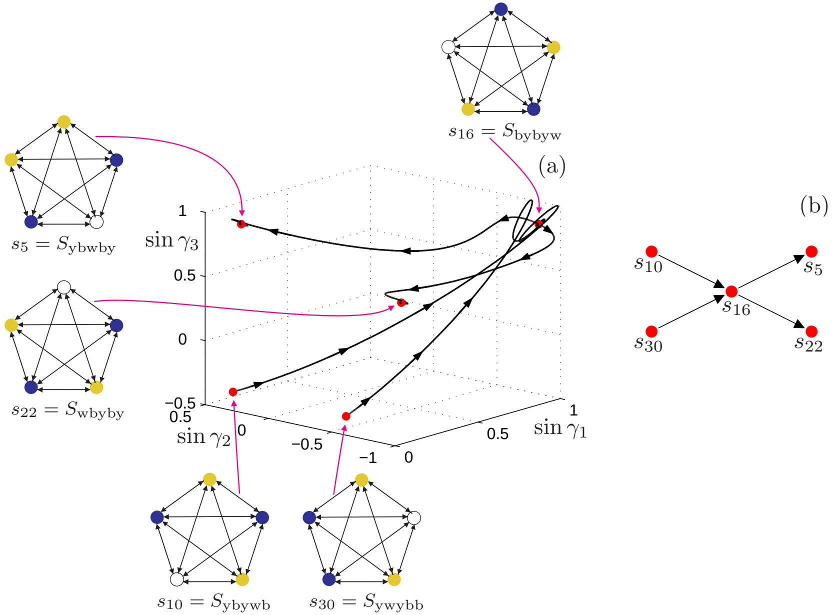

Trajectory of system
Saddle
periodic orbits
Invariant subspaces where
there is phase clustering

Example of transient clustering dynamics:
$N = 5$, $\alpha = 1.8$, $r = 0.2$, $\beta = -2$

$$g(\phi) = -\sin(\phi + \alpha) + r\sin(2\phi + \beta)$$



[P. A., G. Orosz, J. Wordsworth, S. Townley. Reliable switching between cluster states for globally coupled phase oscillators, SIAM J Applied Dynamical Systems **6**:728-758 (2007)]


$s_{16} = S_{\mathrm{bybyw}}$
(a)
$s_5 = S_{\mathrm{ybwby}}$
$s_{22} = S_{\mathrm{wbyby}}$
$s_{10} = S_{\mathrm{ybywb}}$
$s_{30} = S_{\mathrm{ywybb}}$
$\sin\gamma_3$
$\sin\gamma_2$
$\sin\gamma_1$
1
0.5
0
−0.5
−1
(b)
s_{10}
s_5
s_{16}
s_{30}
s_{22}

One can apply this to conductance-based models to find similar dynamics in the presence of synaptic coupling with delays [Karabacak and A, 2009].

Stable clustering

$$\dot{\theta}_i = \omega + \frac{1}{N}\sum_{j=1}^{N} g(\theta_i - \theta_j)\,, \tag{1}$$

Consider M clusters where $1 \leq M \leq N$. Corresponding M-cluster partition $\mathcal{A} = \{A_1, \ldots, A_M\}$ of $\{1, \ldots, N\}$ such that

$$\{1, \ldots, N\} = \bigcup_{p=1}^{M} A_p\,, \tag{2}$$

where A_p are pairwise disjoint sets ($A_p \cap A_q = \emptyset$ if $p \neq q$). NB if $a_p = |A_p|$ then

$$\sum_{p=1}^{M} a_p = N\,. \tag{3}$$

For partition $\mathcal{A}$ associate a subspace

$$\mathbf{T}^N_{\mathcal{A}} = \{\theta \in \mathbf{T}^N : \theta_i = \theta_j \iff \text{there is a } p \text{ such that } i, j \subset A_p\}, \tag{4}$$

and we say a given $\theta \in \mathbf{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$.

Denote phase of the p-th cluster by $\psi_p := \theta_i = \theta_j = \theta_k = \ldots$ for $\{i, j, k, \ldots\} \subset A_p$ we obtain

$$\dot{\psi}_p = \omega + \frac{1}{N} \sum_{q=1}^{M} a_q \, g(\psi_p - \psi_q) \tag{5}$$

for $p = 1, \ldots, M$.
We say $\theta \in \mathbf{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$ *as a periodic orbit* if

$$\psi_p = \Omega \, t + \phi_p \tag{6}$$

for $p = 1, \ldots, M$ and all $\phi_p \pmod{2\pi}$ are different.

Substituting (6) into (5) gives

$$\Omega = \omega + \frac{1}{N}\sum_{q=1}^{M} a_q\, g(\phi_p - \phi_q) \tag{7}$$

for $p = 1, \ldots, M$. By subtracting the last equation ($p = M$) from each of the preceding equations ($p = 1, \ldots, M-1$) we obtain

$$0 = \sum_{q=1}^{M} a_q\, (g(\phi_p - \phi_q) - g(\phi_M - \phi_q)) \tag{8}$$

for $p = 1, \ldots, M-1$. Can determine $M-1$ phases out of ϕ_p, $p = 1, \ldots, M$ while one phase can be chosen arbitrarily, and (7) determines the frequency Ω.

Can compute linear stability in a similar way to above and show [Orosz, A. 2009]:

Theorem

There is a coupling function g for the system (1) such that for any N and any given M-cluster partition $\mathcal{A}$ of $\{1,\ldots,N\}$ there is a linearly stable periodic orbit realizing that partition (and all permutations of it). Moreover, all nearby g in the C^2 norm have a stable periodic orbit with the same partition.

Comments and limitations

In summary, there are practical and numerical ways of reducing and understanding the dynamics of coupled limit cycle oscillators of general type to coupled phase oscillators. This can be useful because:

- Reduces dimension of phase space
- Gives framework for understanding effects of coupling (e.g. pattern formation) on oscillators
- For identical oscillators, can reduce limit cycle problems to equilibrium problems
- Phase dynamics can be highly nontrivial even for quite simple coupling

Tools include:

- Numerical simulation/solution continuation.
- Isochrons/phase response curves/phase transition curves
- Averaging method
- Use of adjoint variational equation
- Analysis of coupled ODEs on a torus
- Studying the synchronization properties
- Symmetric dynamics and bifurcation theory

Limitations

The method of reduction to phase oscillators works well for sufficiently weak couplng, but needs to be treated with respect for:

- Strong coupling
- Weakly attracting/neutrally stable limit cycles
- Chaotic “oscillators”
- Non-smooth systems
- Be careful when averaging in multi-frequency systems

References

K. Josic, E. Brown, J. Moehlis:
http://www.scholarpedia.org/article/Isochron

E. Izhikevich: *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting*, MIT Press, Cambridge, Massachusetts, 2007.
http://www.izhikevich.com

G.B. Ermentrout:
http://www.math.pitt.edu/˜bard/xpp/xpp.html

http://secamlocal.ex.ac.uk/people/staff/pashwin/